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C- and G-parity: A New Definition and Applications —Version VIIb—

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abstract

A new definition for C (charge-conjugation) and G operations is proposed which allows for unique value of the C parity for each member of a given J^{PC} nonet. A simple straightforward extension of the definition allows quarks to be treated on an equal footing. As illustrative examples, the problems of constructing eigenstates of C, I and G operators are worked out for $\pi\pi$, $K\bar{K}$, $N\bar{N}$ and $q\bar{q}$ systems. In particular, a thorough treatment of two-, three- and four-body systems involving $K\bar{K}$ systems is given.

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1 Introduction

The purpose of this note is to point out that the C operation can be defined in such a way that a unique value can be assigned to all the members of a given J^{PC} nonet. In conventional treatments in which antiparticle states are defined through C, one encounters the problem that anti-particle states do not transform in the same way (the so-called charge-conjugate representation). That this is so is obvious if one considers the fact that a C operation changes sign of the z-component of the I-spin, so that in general C and I operators do not commute. In our treatment, the anti-particle states have been defined so that they transform under isospin rotations in exactly the same way as those of particle states. *This allows for use of the same SU(2) Clebsch-Gordan coefficients for particle and anti-particle states, a distinctive advantage in phenomenological applications.*

Arbitrary parameters in the problem have been carefully chosen so as to obtain conventional formulae for CP-eigenstates of neutral kaons K^0 and $\bar{K}^0$. Anti-quark states, conventionally, are defined such that they do not transform in the same way as those quark states. The relationship of our definition to this more conventional one is also pointed out.

As illustrative examples, states of particle-antiparticle systems are worked out, in which use is made of the symmetry under interchange of two wave functions (spin statistics). A treatment of two-, three- and four-body channels which contain $K\bar{K}$ states, i.e. $K\bar{K}$ (at the end of Section 4 and Section 5), $K\bar{K}\pi$ (Section 6), $K\bar{K}\pi\pi$ (Section 7), is worked out in some detail, in particular to point out several non-trivial results regarding the system.

The appendix is devoted to an exposition of the systematics concerning CP-violating parameters ϵ_S and ϵ_L for the neutral K mesons K_S and K_L.

2 G-and C-parity Operations

We shall adopt a notation 'a' to stand for both the baryon number B and hypercharge $Y = B + S$. Anti-particles are denoted '$\bar{a}$', so that

$$a = (B, Y), \quad \bar{a} = (\bar{B}, \bar{Y}) = (-B, -Y) \tag{1}$$

In addition, we shall use y to denote $Y/2$;

$$y = \frac{Y}{2} = \frac{1}{2}(B + S), \quad \boxed{Q = y + \nu} \tag{2}$$

where S, Q, ν are the strangeness, the charge and the third component of isospin, respectively.

We start with a state having an isospin σ and its third component ν which transforms according to the standard $|jm\rangle$ representation. Let I be the isospin operator. Then,

$$\boxed{\begin{aligned} I_z|\sigma\nu\rangle &= \nu|\sigma\nu\rangle \\ I_\pm|\sigma\nu\rangle &= F_\pm(\nu)|\sigma\nu \pm 1\rangle \\ I^2|\sigma\nu\rangle &= \sigma(\sigma+1)|\sigma\nu\rangle \end{aligned}} \tag{3}$$

where $I_\pm$=$I_x$$\pm$ iI_y and

$$F_\pm(\nu) = \sqrt{(\sigma \mp \nu)(\sigma \pm \nu + 1)} \tag{4}$$

Note that $F_\pm(\nu) = F_\mp(\ -\nu)$. We shall require that anti-particle states transform in the same way as their particle states according to (3).

The C operation changes a state $|a\nu\rangle$ to $|\bar{a}\ -\nu\rangle$. (We use a shorthand notation where the isospin σ is omitted from a more complete description of the state $|a\sigma\nu\rangle$.) If anti-particle states are to transform in the same way as particle states, it is necessary that one define an anti-particle through the G operation. The key point is that G is defined so that its operation does not perturb the ν quantum number. To define the G operator, we need to first introduce a rotation by 180° around the y-axis:

$$R_y(\pi)|\sigma\nu\rangle = (-)^{\sigma-\nu}|\sigma\ \ -\nu\rangle \tag{5}$$

It will be shown later that $R_y(\pi)$ commutes with the C operator. We therefore define the G operator through

$$G = CR_y(\pi) = R_y(\pi)C \tag{6}$$

We are now ready to define an anti-particle state via

$$\boxed{\begin{aligned} G|a\nu\rangle &= g|\bar{a}\nu\rangle \\ G|\bar{a}\nu\rangle &= \bar{g}|a\nu\rangle \end{aligned}} \tag{7}$$

and *require* that g and $\bar{g}$ be independent of ν and furthermore that an arbitrary isospin rotation R commutes with G:

$$[R, G] = 0 \tag{8}$$

The action of C on particle and anti-particle states is readily discerned through (5) and (6);

$$\begin{aligned} C|a\nu\rangle &= g(-)^{\sigma+\nu}|\bar{a}\ -\nu\rangle \\ C|\bar{a}\nu\rangle &= \bar{g}(-)^{\sigma+\nu}|a\ -\nu\rangle \end{aligned} \tag{9}$$

It is customary to define C such that $C^2 = +\ 1$, in which case

$$g\bar{g}(-)^{2\sigma} = +1 \tag{10}$$

For hadrons, we shall define g and $\bar{g}$ via

$$g = \eta(-)^{y+\sigma}, \quad \bar{g} = \eta(-)^{\bar{y}+\sigma} \tag{11}$$

while for quarks,

$$g = \eta(-)^{B+y+\sigma}, \quad \bar{g} = \eta(-)^{\bar{B}+\bar{y}+\sigma} \tag{12}$$

Note that the exponents in these expressions are always integers. Note in particular that, for quarks, B and $\bar{B}$ are needed to make the exponents integers. The quantity F defined by

$$F = 2B + Y \quad \text{and} \quad f = \frac{1}{2}F = B + y = \frac{1}{2}(3B + S) \tag{13}$$

will be termed the '*intrinsic flavor*' of a particle. Note that the intrinsic flavor, together with S and $2I$ which characterize a particle, are always integers, as shown in the following table:

states	$\begin{pmatrix} u \\ d \end{pmatrix}$	s	π	η	K	$\begin{pmatrix} p \\ n \end{pmatrix}$	Λ	Σ	Ξ	Δ	Ω^-
F	1	0	0	0	1	3	2	2	1	3	0
S	0	-1	0	0	$+1$	0	-1	-1	-2	0	-3
$2I$	1	0	2	0	1	1	0	2	1	3	0

It is seen that the intrinsic flavor of an anti-particle is the negative of that of the particle, i.e. $\bar{F} = -F$. With these definitions, we can make η a real number and let it take on values of $+1$ or-1, so that $\eta^2 = +1$. Then, we have,

$$C^2 = +1, \quad G^2 = (-)^{2\sigma} \tag{14}$$

conforming to the standard expressions. An identity of a particle and its antiparticle partner can now be expressed by, instead of (1),

$$b = (3B, S), \quad \bar{b} = (3\bar{B}, \bar{S}) = (-3B, -S), \quad f = \frac{1}{2}F = \frac{1}{2}(3B + S)$$

So a particle and its antiparticle partner can be identified by either $a = (B, Y)$ or by $b = (3B, S)$. The latter has the advantage that the arguments of b are always integers for both quarks and particles. Its sum $F = 3B + S$ (the intrinsic flavor) and S, which may be grouped together in $c = (F, S)$, may also be used to designate a particle and its antiparticle partner. Both F and S are integers for all particles, *including quarks*, and $F \geq 0$ for all particles (so antiparticles come with $\bar{F} \leq 0$). For example, the s quark is characterized by $(F, S) = (0, -1)$ and $\bar{s}$ by $(\bar{F}, \bar{S}) = (0, +1)$, while Ω^- is denoted by $(F, S) = (0, -3)$ and $\bar{\Omega}^+$ by $(\bar{F}, \bar{S}) = (0, +3)$. To take one final example, consider the doublet $\{u, d\}$; it is characterized by $(F, S) = (1, 0)$, whereas its antiparticle counterpart $\{\bar{u}, \bar{d}\}$ is given by $(\bar{F}, \bar{S}) = (-1, 0)$.

We have adopted two different definitions for g—one for hadrons and another for quarks; however, we could have chosen a single convention in which the expression (12) is used for both hadrons and for quarks. The factor $(-)^B$ for mesons is +1, while it is -1 for baryons, but the extra minus sign could be simply absorbed into η. Although it might be aesthetically pleasing to have a single definition for g (along with the 'intrinsic flavor' F defined above), we choose to opt for more transparent definitions.

From (3) and (9), it is easy to work out the commutation relations between C and I;

$$\{C, I_x\} = \{C, I_z\} = 0, \quad [C, I_y] = 0 \tag{15}$$

ln other words, C anti-commutes with I_x and I_z while it commutes with I_y. This gives a ready justification of the definition of G-parity given in (6). From (8), we can further deduce that

$$CRC^{-1} = R_y(\pi) R R_y^{-1}(\pi) \tag{16}$$

This shows that the actions of I-spin rotation under charge-conjugation can be expressed in terms of I-spin 180° rotations.

We end this section by recapitulating the actions of G and C. For hadrons, we have, from (7), (9) and (11),

$$\boxed{\begin{aligned} G|a\nu\rangle &= \eta(-)^{y+\sigma}|\bar{a}\nu\rangle \\ G|\bar{a}\nu\rangle &= \eta(-)^{\bar{y}+\sigma}|a\nu\rangle \end{aligned}} \tag{17}$$

also

$$\boxed{\begin{aligned} C|a\nu\rangle &= \eta(-)^{y-\nu}|\bar{a}\ \ -\nu\rangle \\ C|\bar{a}\nu\rangle &= \eta(-)^{\bar{y}-\nu}|a\ \ -\nu\rangle \end{aligned}} \tag{18}$$

and, likewise, for quarks we have,

$$\begin{aligned} G|a\nu\rangle &= \eta(-)^{B+y+\sigma}|\bar{a}\nu\rangle \\ G|\bar{a}\nu\rangle &= \eta(-)^{\bar{B}+\bar{y}+\sigma}|a\nu\rangle \end{aligned} \tag{19}$$

$$\begin{aligned} C|a\nu\rangle &= \eta(-)^{B+y-\nu}|\bar{a}\ \ -\nu\rangle \\ C|\bar{a}\nu\rangle &= \eta(-)^{\bar{B}+\bar{y}-\nu}|a\ \ -\nu\rangle \end{aligned} \tag{20}$$

Note that $\bar{y} = -y$ and $\bar{B} = -B$. It is worth emphasizing again that all the exponents in (17) through (20) are integers. Note in addition that only the u and d quarks have

I-spin flavors under strong interactions. So, in practice, (19) and (20) apply only to these quarks, although s and other heavier quarks, c, b and t, can also be described in the same fashion with $\sigma = 0$ and the strangeness S replaced by heavier flavors $C(= +1)$, $B(= -1)$ and $T(= +1)$. Here B stands for the "Bottom" flavor and *not* the baryon number B, a notation we use elsewhere in this note.

We come back to the concept of the intrinsic flavor $F = 2f$ of a particle, in order to come up with a uniform description of quarks and particles. Replacing $a = (B, Y)$ by $b = (3B, S)$ in the ket states, we can write

$$
\begin{aligned}
G|\, b\,\nu\rangle &= \eta(-)^{f+\sigma}|\,\bar{b}\,\nu\rangle, &\qquad G|\,\bar{b}\nu\rangle &= \eta(-)^{\bar{f}+\sigma}|\,b\,\nu\rangle \\
C|\, b\,\nu\rangle &= \eta(-)^{f-\nu}|\,\bar{b}\;\; -\nu\rangle, &\qquad C|\,\bar{b}\nu\rangle &= \eta(-)^{\bar{f}-\nu}|\,b\;\; -\nu\rangle
\end{aligned}
$$

It may be worth repeating once more: the intrinsic flavor F is a non-negative integer for any quarks or elementary particles. The species of particles covered in this note, including quarks, are uniquely specified by the integer "quantum numbers" $\{F, S, 2I\}$, where $F = B + Y$ [see (13)]. See the preceding table for a list of quarks and particles.

3 Examples of Single-Particle States

Nonstrange neutral mesons have, of course, $a = \bar{a}$, $y = \bar{y} = 0$ and $\nu = 0$. The expression (18) shows that a state $|a\nu\rangle$ is in an eigenstate of C with the eigenvalue η. Nonstrange charged or neutral mesons are, according to (17), in an eigenstate of G with the eigenvalue $\eta(-)^\sigma$, a familiar result. Since the expressions (17) and (18) are general and applies to all hadrons, it is natural to extend to the strange members of a given J^{PC} nonet the same η which is determined for only the nonstrange neutral member of the family. It should be borne in mind, however, that the value of η cannot be directly determined for strange mesons.

As a first example, let us take the π nonet ($J^{PC} = 0^{-+}$) and exhibit the actions G and

C. Here $\eta = +1$ and so the actions of C and G for all the members of the nonet are[a]

$$\begin{gathered} G\,\pi = -\pi, \qquad \text{and} \qquad G\,\eta = +\eta \\ C\,\pi^{\pm} = -\pi^{\mp}, \qquad C\,\pi^0 = +\pi^0, \qquad \text{and} \qquad C\,\eta = +\eta \end{gathered} \tag{21}$$

and

$$\begin{aligned} G\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} &= \begin{pmatrix} -\bar{K}^0 \\ -K^- \end{pmatrix}, \qquad G\begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix} = \begin{pmatrix} +K^+ \\ +K^0 \end{pmatrix} \\ C\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} &= \begin{pmatrix} +K^- \\ -\bar{K}^0 \end{pmatrix}, \qquad C\begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix} = \begin{pmatrix} -K^0 \\ +K^+ \end{pmatrix} \end{aligned} \tag{22}$$

The eigenstates of CP for a neutral K system is now given by the familiar relations.

$$\begin{cases} |K_1\rangle = \dfrac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle) \\ |K_2\rangle = \dfrac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle) \end{cases} \tag{23}$$

where the physically-allowed neutral K states K_S and K_L are nearly equal to K_1 and K_2, respectively—but not identical—due to CP-violation in their decays (see Appendix). The actions of G and C operators on the ρ nonet can be obtained by replacing the particles in (21) and (22) by those of the ρ nonet and reversing all the signs, since the parameter η of the previous section is now -1 for this nonet.

The nonet of ground-state baryons containing nucleons can be defined in a similar way as in (21). Since the neutral members are no longer in an eigenstate of C or G, we arbitrarily set $\eta = +1$ for this nonet (or $\eta = -1$ if the intrinsic flavor is used):

$$G|p\rangle = -|\bar{n}\rangle, \qquad C|p\rangle = +|\bar{p}\rangle \tag{24}$$

$$\begin{aligned} G|n\rangle &= -|\bar{p}\rangle, \qquad C|n\rangle = -|\bar{n}\rangle \\ G|\bar{n}\rangle &= +|p\rangle, \qquad C|\bar{n}\rangle = -|n\rangle \\ G|\bar{p}\rangle &= +|n\rangle, \qquad C|\bar{p}\rangle = +|p\rangle \end{aligned} \tag{25}$$

Other members of the nonet, Λ, Ξ, Σ, can be worked out in a similar way.

[a] The parameter η used in the previous section should not be confused with the particle η of the π nonet.

From (19) and (20), it can be seen that the up- and down-quark states will transform in exactly the same way as in (24) and (25) if we again set $\eta = +1$; merely replace p and n in these formulae by u and d. We exhibit isospin 0 and 1 states for quark-antiquark systems:

$$
\begin{aligned}
|00\rangle &= \frac{1}{\sqrt{2}}\Big(|u\rangle|\bar{u}\rangle - |d\rangle|\bar{d}\rangle\Big) \\
|10\rangle &= \frac{1}{\sqrt{2}}\Big(|u\rangle|\bar{u}\rangle + |d\rangle|\bar{d}\rangle\Big)
\end{aligned}
\tag{26}
$$

where we have used the usual Clebsch-Gordon Coefficient tables. However, it has now become customary to define antiquark states via C operations as follows,

$$
\begin{aligned}
C|u\rangle &= +|\bar{u}\rangle_c, \qquad & C|d\rangle &= +|\bar{d}\rangle_c \\
C|\bar{d}\rangle &= +|d\rangle_c, \qquad & C|\bar{u}\rangle &= +|u\rangle_c
\end{aligned}
\tag{27}
$$

where the subscripts c have been introduced to distinguish these from our definition of antiparticle states as given in (24) and (25). It is clear that the up-quark states transform in the same way, but the down-quark states differ by a minus sign. In terms of the more conventional definitions (27), the isospin 0 and 1 now read

$$
\begin{aligned}
|00\rangle &= \frac{1}{\sqrt{2}}\Big(|u\rangle_c|\bar{u}\rangle_c + |d\rangle_c|\bar{d}\rangle_c\Big) \\
|10\rangle &= \frac{1}{\sqrt{2}}\Big(|u\rangle_c|\bar{u}\rangle_c - |d\rangle_c|\bar{d}\rangle_c\Big)
\end{aligned}
\tag{28}
$$

It is in these forms that one encounters most often the isospin states of quark-antiquark systems. One must not forget, however, that these antiquark states do not have the 'correct' isospin rotation properties.

4 Two-Particle States

We shall work out here the effect of C and G operations on a particle-antiparticle system in an eigenstate of total isospin, total intrinsic spin, orbital angular momentum and total spin. We use the notations I, S, ℓ and J for these quantum numbers. (Note that I was used as an isospin operator and S denoted strangeness in Section 2.) Each single-particle state in the two-particle center-of-mass(CM) system will be given a shorthand notation,

$$
\begin{aligned}
|a, +\vec{k}, \nu_1, m_1\rangle &= |a, +\vec{k}, \sigma_1\nu_1, s_1 m_1\rangle \\
|\bar{a}, -\vec{k}, \nu_2, m_2\rangle &= |\bar{a}, -\vec{k}, \sigma_2\nu_2, s_2 m_2\rangle
\end{aligned}
\tag{29}
$$

where $\vec{k}$ is the 3-momentum of the particle in the CM system, and σ_1 and s_1 are isospin and spin of the particles $\sigma_1 = \sigma_2 = \sigma$ and $s_1 = s_2 = s$.

The two-particle system in a given state of $|I\nu\rangle$ and $|\ell SJM\rangle$ is given by

$$
\begin{aligned}
|a\bar{a}\nu\rangle = \sum_{\substack{\nu_1\,\nu_2 \\ m_1\,m_2}} &(\sigma_1\nu_1\sigma_2\nu_2|I\nu)(s_1 m_1 s_2 m_2|S m_s)(S m_s \ell m|JM) \\
&\times \int \mathrm{d}\vec{k}\; Y^\ell_m(\vec{k})\; |a, +\vec{k}, \nu_1, m_1\rangle |\bar{a}, -\vec{k}, \nu_2, m_2\rangle
\end{aligned}
\tag{30}
$$

where $Y^\ell_m(\vec{k})$ is the usual spherical harmonics. We note, from (17) through (20),

$$
\begin{aligned}
C|a, +\vec{k}, \nu_1, m_1\rangle|\bar{a}, -\vec{k}, \nu_2, m_2\rangle &= (-)^{\nu_1+\nu_2}|\bar{a}, +\vec{k}, -\nu_1, m_1\rangle|a, -\vec{k}, \;-\nu_2, m_2\rangle \\
&= (-)^{\nu_1+\nu_2+2s}|a, -\vec{k}, \;-\nu_2, m_2\rangle|\bar{a}, +\vec{k}, \;-\nu_1, m_1\rangle \\
G|a, +\vec{k}, \nu_1, m_1\rangle|\bar{a}, -\vec{k}, \nu_2, m_2\rangle &= (-)^{2\sigma}|\bar{a}, +\vec{k}, \nu_1, m_1\rangle|a, -\vec{k}, \nu_2, m_2\rangle \\
&= (-)^{2\sigma+2s}|a, -\vec{k}, \nu_2, m_2\rangle|\bar{a}, +\vec{k}, \nu_1, m_1\rangle
\end{aligned}
\tag{31}
$$

where the second lines have been derived by interchanging two wave functions, which brings in a factor $(-)^{2s}$, positive for mesons and negative for fermions.

The effect of C and G on the two-particle states can now be worked out. By interchanging the subscripts 1 and 2 and by the operation $\vec{k} \to -\vec{k}$, we obtain

$$
\boxed{
\begin{aligned}
C|a\bar{a}\nu\rangle &= (-)^{\ell+S+\nu}\; |a\bar{a}\; -\nu\rangle \qquad (32)\\
G|a\bar{a}\nu\rangle &= (-)^{\ell+S+I}\; |a\bar{a}\nu\rangle \qquad (33)
\end{aligned}
}
$$

where we have used the relationship

$$
Y^\ell_m(-\vec{k}) = (-)^\ell\, Y^\ell_m(\vec{k})
$$

and the following formulas for the Clebsch-Gordan coefficient

$$(\sigma_2 - \nu_2 \sigma_1 - \nu_1 | I\nu) = (\sigma_1 \nu_1 \sigma_2 \nu_2 | I\nu)$$

$$(\sigma_2 \nu_2 \sigma_1 \nu_1 | I\nu) = (-)^{I-2\sigma} (\sigma_1 \nu_1 \sigma_2 \nu_2 | I\nu), \quad \sigma_1 = \sigma_2 = \sigma$$

$$(s_2 m_2 s_1 m_1 | S m_s) = (-)^{S-2s} (s_1 m_1 s_2 m_2 | S m_s), \quad s_1 = s_2 = s$$

For completeness, we also work out the effect of the parity operation(Π) on the two-particle states. Since antifermions have opposite intrinsic parities to those of their fermion partners, the Π operation brings in the factor $(-)^{2s}$. In addition, the 3-momentum $\vec{k}$ changes sign under the Π operation. Therefore, we have

$$\Pi|a, +\vec{k}, \nu_1, m_1\rangle|\bar{a}, -\vec{k}, \nu_2, m_2\rangle = (-)^{2s}|a, -\vec{k}, \nu_1, m_1\rangle|\bar{a}, +\vec{k}, \nu_2, m_2\rangle \tag{34}$$

So, again by using the operation $\vec{k} \to -\vec{k}$, we obtain the familiar result

$$\boxed{\Pi|a\bar{a}\nu\rangle = (-)^{\ell+2s}\ |a\bar{a}\nu\rangle} \tag{35}$$

It follows from (32) that a particle-antiparticle with $\nu = 0$ is in an eigenstate of C with its eigenvalue $(-)^{\ell+S}$. This result applies to all neutral $N\bar{N}$, $q\bar{q}$, $K\bar{K}$ and $\pi\pi$ systems, with $S = 0$ for dikaon and dipion systems. For all ν, a particle-antiparticle system has the G-parity equal to $(-)^{\ell+S+I}$ [see (33)]. Charged $N\bar{N}$, $q\bar{q}$, $K\bar{K}$ systems have $I = 1$, so that their G-parity is $(-)^{\ell+S+1}$ (again $S = 0$ for dikaons). Since the G-parity is $+1$ for dipions, one has

$$I + \ell = \text{ even}$$

for any $\pi\pi$ system. For all ν, the intrinsic parity of a particle-antiparticle system is given by $(-)^{\ell+2s}$ [see (35)].

We next apply the formulas (32) and (33) to $K\bar{K}$ systems (see the next section for a more complete treatment). We see that its G-parity is given by

$$g = (-)^{\ell+I}$$

while its C-parity is simply given by $(-)^{\ell}$. We see that $I + \ell =$ must be *even* if $g = +1$. There are four outstanding examples: (1) the first example with $I = \ell = 0$ is the $f_0(980)$,

$I^G\,(J^{PC}) = 0^+\,(0^{++})$, which is thought to be a $K\bar{K}$ molecule; (2) the second example with $I=0$ and $\ell=2$ is the $f_2'(1525)$ with $I^G\,(J^{PC}) = 0^+\,(2^{++})$, which is known to decay into $K\bar{K}$ with a branching ratio of $(88.7 \pm 2.2)\%$; (3) the third example with $I=\ell=1$ is the $s\bar{s}$ component of the $\rho(1700)$, a $K\bar{K}$ state in a P-wave. We should note that the $\rho(1700)$ also decays into $K\bar{K}^*(890) \oplus \bar{K}\,K^*(890)$. See Section 6 for a thorough analysis of the $K\bar{K}\pi \oplus \bar{K}K\pi$; (4) finally, if $g=-1$, then

$$I + \ell = \text{odd}$$

and the $a_2(1320)$ with $I^G\,(J^{PC}) = 1^-\,(2^{++})$ becomes an outstanding example; here $I=1$ and $\ell=2$.

We now explore the consequences of the Bose-Fermi symmetry when two particles are identical, i.e. a and $\bar{a}$ are now identical particles, i.e. $a=\bar{a}$, $\sigma_1=\sigma_2=\sigma$ and $s_1=s_2=s$. We switch the two ket states as given in (30) and find, taking into account of the spin statistics,

$$\begin{aligned}
|\bar{a}a\nu\rangle_s &= (-)^{2s} \sum_{\substack{\nu_1\,\nu_2\\ m_1\,m_2}} (\sigma_1\nu_1\sigma_2\nu_2|I\nu)(s_1m_1s_2m_2|Sm_s)(Sm_s\ell m|JM) \\
&\quad \times \int \mathrm{d}\vec{k}\; Y_m^\ell(\vec{k})\; |\bar{a},-\vec{k},\nu_2,m_2\rangle|a,+\vec{k},\nu_1,m_1\rangle \qquad (36a)\\
&= (-)^{2s} \sum_{\substack{\nu_1\,\nu_2\\ m_1\,m_2}} (\sigma_1\nu_1\sigma_2\nu_2|I\nu)(s_1m_1s_2m_2|Sm_s)(Sm_s\ell m|JM) \\
(\vec{k}\to-\vec{k}) \to &\quad \times (-)^\ell \int \mathrm{d}\vec{k}\; Y_m^\ell(\vec{k})\; |\bar{a},+\vec{k},\nu_2,m_2\rangle|a,-\vec{k},\nu_1,m_1\rangle \\
(1\leftrightarrow 2) \to &= (-)^{2s} \sum_{\substack{\nu_1\,\nu_2\\ m_1\,m_2}} (\sigma_2\nu_2\sigma_1\nu_1|I\nu)(s_2m_2s_1m_1|Sm_s)(Sm_s\ell m|JM) \\
&\quad \times (-)^\ell \int \mathrm{d}\vec{k}\; Y_m^\ell(\vec{k})\; |\bar{a},+\vec{k},\nu_1,m_1\rangle|a,-\vec{k},\nu_2,m_2\rangle \qquad (36b)\\
&= (-)^{2s}(-)^{\sigma_1+\sigma_2-I}\,(-)^{s_1+s_2-S} \sum_{\substack{\nu_1\,\nu_2\\ m_1\,m_2}} (\sigma_1\nu_1\sigma_2\nu_2|I\nu)(s_1m_1s_2m_2|Sm_s) \\
&\quad \times (-)^\ell (Sm_s\ell m|JM) \int \mathrm{d}\vec{k}\; Y_m^\ell(\vec{k})\; |\bar{a},+\vec{k},\nu_1,m_1\rangle|a,-\vec{k},\nu_2,m_2\rangle \qquad (36c)
\end{aligned}$$

Noting that ℓ, $\sigma_1+\sigma_2=2\sigma$ and $s_1+s_2=2s$ are integers, if we now require

$$\boxed{(-)^{I+S+\ell} = (-)^{2\sigma}} \qquad (37)$$

then (30) and (36) become identical—which simply expresses the requirement that the Bose-Fermi symmetry holds. We emphasize that (37) applies to both fermions and mesons. It simply states that $I + S + \ell$ is even (odd) if σ is an integer (odd-half integer). As an example, we apply (37) to a dipion system, and find that we must demand $I + \ell =$ even; this result has already been obtained from a study of the G-parity of a dipion system, as noted in the previous paragraph. As a second example, we apply (37) to a $\phi\phi$ system. Here $I = 0$ and so we must require $S + \ell =$ even, i.e. $\ell =$ odd if $S = 1$ and $\ell =$ even if $S = 0$ or 2. As another example, consider a $\rho^0\rho^0$ system. Here we must have $I = 0$ (or 2), so that $S + \ell =$ even; and hence the conclusions are the same as in the $\phi\phi$ system. Consider next a dibaryon system, NN, where $N = \{p, n\}$, which must satisfy

$$I + S + \ell = \text{odd} \qquad \text{(for a dibaryon } NN \text{ system)} \tag{38}$$

If $I = 0$, then $S + \ell =$ odd, whereas if $I = 1$, then we must require $S + \ell =$ even. As a final example, consider a diquark system qq with $q = \{u, d\}$ in a baryon. Since the color index for the three quarks must be completely antisymmetric to form a color singlet baryon, the diquark system must be symmetric under interchange of the two quarks in spin-isospin space, i.e.

$$I + S + \ell = \text{even} \qquad \text{(for a diquark } qq \text{ system in a baryon)} \tag{39}$$

which is obtained by adding a factor $(-)^{2s}$ to the equation (37). Or, equivalently, by dropping the same factor from the equations (36a), (36b) and (36c). The ground-state qq system must have $\ell = 0$, so that $I + S =$ even. This means that the qq states come either in $I = S = 0$ with $J^P = 0^+$ leading to p and n, or they can come in $I = S = 1$ with $J^P = 1^+$, which leads not only to the $N(1440)$ ($I = \frac{1}{2}$ and $J^P = \frac{1}{2}^+$) but also to the $\Delta(1232)$ ($I = \frac{3}{2}$ and $J^P = \frac{3}{2}^+$).

Consider now an $s\bar{s}$ quarkonium and apply the formula (37). We see that $\sigma = 1/2$ and $I = 0$, so that we must have $S + \ell =$ odd where $S = 0$ or 1. If $\ell = 0$ so that $S = 1$, then we have the $s\bar{s}$ component of the pseudo-vector nonet, i.e. the $h_1'(1170?)$. If $\ell = 1$, then $S = 0$ and so we obtain the $s\bar{s}$ component of the vector nonet, i.e. the $\phi(1020)$.

5 Two-Particle State: A $K\bar{K}$ System

We now turn to the task of describing the $K\bar{K}$ System. It should be noted, once again, that the expression (37) holds only when $a = \bar{a}$; so it does not apply to $|K\bar{K}\rangle$, since it is not equal to $|\bar{K}K\rangle$. We set

$$|I,\, K\bar{K}\,\rangle = \frac{1}{\sqrt{2}}\Big(|K^+\, K^-\rangle - (-)^I|K^0\, \bar{K}^0\,\rangle\Big), \quad I = 0 \ \text{ or } \ 1 \tag{40}$$

and

$$|I,\, \bar{K}\, K\,\rangle = \frac{1}{\sqrt{2}}\Big(|\bar{K}^0\, K^0\,\rangle - (-)^I|K^-\, K^+\,\rangle\Big), \quad I = 0 \ \text{ or } \ 1 \tag{41}$$

Note that, from (22),

$$\begin{aligned} G\,|I,\, K\,\bar{K}\,\rangle &= \frac{1}{\sqrt{2}}\Big(-|\bar{K}^0\, K^0\,\rangle + (-)^I|K^-\, K^+\,\rangle\Big) = -|I,\, \bar{K}\, K\,\rangle, \quad I = 0 \ \text{ or } \ 1 \\ G\,|I,\, \bar{K}\, K\,\rangle &= \frac{1}{\sqrt{2}}\Big(-|\bar{K}^+\, K^-\,\rangle + (-)^I|K^0\, \bar{K}^0\,\rangle\Big) = -|I,\, K\bar{K}\,\rangle, \quad I = 0 \ \text{ or } \ 1 \end{aligned} \tag{42}$$

The combined wave function in an eigenstate of G-parity with an eigenvalue g is, applying a proper normalization constant,[b]

$$|^{(0)}\Psi_I^g\,\rangle = \frac{1}{2}\Big(|I,\, K\bar{K}\,\rangle - g\,|I,\, \bar{K}\, K\,\rangle\Big) \tag{43}$$

or

$$\begin{aligned} |^{(0)}\Psi_I^g\,\rangle = \frac{1}{2\sqrt{2}}&\Big(|K^+\, K^-\rangle + g(-)^I|K^-\, K^+\,\rangle\Big) \\ &- \frac{1}{2\sqrt{2}}(-)^I\Big(|K^0\, \bar{K}^0\,\rangle + g(-)^I|\bar{K}^0\, K^0\,\rangle\Big) \end{aligned} \tag{44}$$

so that

$$\boxed{G\,|^{(0)}\Psi_I^g\,\rangle = g\,|^{(0)}\Psi_I^g\,\rangle, \qquad C\,|^{(0)}\Psi_I^g\,\rangle = g(-)^I\,|^{(0)}\Psi_I^g\,\rangle} \tag{45}$$

This state is, from (32) and (33), in an eigenstate of G and C with its eigenvalue $(-)^{\ell+I}$ and $(-)^\ell$. So the C eigenvalue is independent of I.

---

[b] This is necessary so that the resulting wave function [see (46*b*)] confirms to the normalization given in (40). The basic reason for this is that the equations (40) and (41) are equal to each other for a properly defined g.

We can further reduce the expression (44) by switching the K's in the second terms of the parentheses above and obtain

$$
\begin{aligned}
|{}^{(0)}\Psi_I^g\rangle &= \frac{1}{2\sqrt{2}}\Big(1+g(-)^{\ell+I}\Big)\Big(|K^+K^-\rangle - (-)^I|K^0\bar{K}^0\rangle\Big) && (46a)\\
&= \frac{1}{\sqrt{2}}\Big(|K^+K^-\rangle - (-)^I|K^0\bar{K}^0\rangle\Big), \qquad g=(-)^{\ell+I} && (46b)\\
(84)\rightarrow\ &= \frac{1}{\sqrt{2}}|K^+K^-\rangle - \frac{(-)^I}{2\sqrt{2}}\Big(|K_1K_1\rangle - |K_2K_2\rangle \\
&\qquad - |K_1K_2\rangle + |K_2K_1\rangle\Big) && (46c)\\
(94)\rightarrow\ &= \frac{1}{\sqrt{2}}|K^+K^-\rangle - \frac{(-)^I}{2\sqrt{2}}\cdot\frac{1+|\epsilon|^2}{1-\epsilon^2}\Big(|K_SK_S\rangle - |K_LK_L\rangle \\
&\qquad - |K_SK_L\rangle + |K_LK_S\rangle\Big) && (46d)\\
&= \frac{1}{\sqrt{2}}|K^+K^-\rangle - \frac{(-)^I}{2\sqrt{2}}\cdot\frac{1+|\epsilon|^2}{1-\epsilon^2}\Bigg[\Big(|K_SK_S\rangle - |K_LK_L\rangle\Big) \\
&\qquad - \frac{1}{\sqrt{2}}\left(1-(-)^\ell\right)|K_SK_L\rangle\Bigg] && (46e)
\end{aligned}
$$

Note that (46b) above is identical to (40), confirming self-consistency in the normalization chosen in (43). A comment is in order regarding the coefficient in front of the term $|K_S K_S\rangle$. Its magnitude is slightly bigger than $1/(2\sqrt{2})$ and it is complex. Since the final states $|K^+K^-\rangle$ and $|K_S K_S\rangle$ cannot be simultaneously measured, the complex phase of the coefficient cannot be measured in the strong-interaction reactions. The absolute square of the coefficient merely indicates the relative strength of the final states $|K^+K^-\rangle$ and $|K_S K_S\rangle$. The spin (or ℓ) of the $|K_S K_S\rangle$ (as well as the $|K_L K_L\rangle$) must be even because of the Bose symmetry and so its C-parity is equal to $+1$ and its G-parity is determined by the isospin, i.e. $g = (-)^I$. The decay mode $a_2^0(1320) \rightarrow K_S K_S$, with a branching ratio[c] of $[(4.9 \pm 0.8)/8]\,\%$, provides a well-known example; note that the quantum numbers $I^G\,(J^{PC}) = 1^-\,(2^{++})$ of the $a_2(1320)$ meson are consistent with this decay mode. It is clear that an odd-ℓ $K\bar{K}$ systems must couple to $|K_S K_L\rangle$ only with its C-parity equal to -1 and its G-parity given by $g = (-)^{I+1}$. The most prominent decay of this type is $\phi(1020) \rightarrow K_S K_L$ which has a branching ratio of $(34.2 \pm 0.4)\%$. The equation (46d) predicts that the $\phi(1020)$ branching ratio into K^+K^- is 50%, while its branching ratio into $K_S K_L$ is 25% (exactly, if $\epsilon = 0$),[d] whereas the PDG (July 2010)

[c] The factor 1/8 comes from the equation (46d) for which we set $\epsilon = 0$.

cites the values of 48.9% and 34.2% for them. Evidently, higher-order loop diagrams in the decay process contribute to an enhanced branching ratio into $K_S\,K_L$. A summary of the neutral $K\bar{K}$ systems is given in the following table:

Distinct Final States for Neutral $K\bar{K}$ Systems[a] for J up to 6
[Eigenstates of C with its eigenvalue $(-)^\ell$ which is indendent of I]

Final State	BR	(Allowed I^G's)	(Allowed (J^{PC}'s)
$K^+\,K^-$	1/2	$(0^+,\,1^-)$	$(0^{++}),\ (2^{++}),\ (4^{++}),\ (6^{++})$
		$(0^-,\,1^+)$	$(1^{--}),\ (3^{--}),\ (5^{--})$
$K_S\,K_S$[b]	1/8[c]	$(0^+,\,1^-)$	$(0^{++}),\ (2^{++}),\ (4^{++}),\ (6^{++})$
$K_L\,K_L$[d]	1/8	$(0^+,\,1^-)$	$(0^{++}),\ (2^{++}),\ (4^{++}),\ (6^{++})$
$K_S\,K_L$	1/4	$(0^-,\,1^+)$	$(1^{--}),\ (3^{--}),\ (5^{--})$

[a] Based on (46) with $\epsilon = 0$.
[b] Prime channel for detection, because $c\tau = 2.7\,\mathrm{cm}$ (see Appendix).
[c] The branching ratio for both K_S's to decay via $K_S \to \pi^+\pi^-$ is $(1/8)\times(0.692)^2$.
[d] Impractical to observe, because $c\tau = 1\,534\,\mathrm{cm}$ (see Appendix).

A charged $K\bar{K}$ system with its net strangeness zero is necessarily I=1, so that the wave functions for $Q = \pm 1$ are

$$\begin{aligned} |{}^{(+)}\Psi_1^g\rangle &= \frac{1}{2}\Big(|K^+\,\bar{K}^0\rangle - g\,|\bar{K}^0\,K^+\rangle\Big) = \frac{1}{2}\Big(1 - g(-)^\ell\Big)\,|K^+\,\bar{K}^0\rangle \\ |{}^{(-)}\Psi_1^g\rangle &= \frac{1}{2}\Big(|K^0\,K^-\rangle - g\,|K^-\,K^0\rangle\Big) = \frac{1}{2}\Big(1 - g(-)^\ell\Big)\,|K^0\,K^-\rangle \end{aligned} \tag{47}$$

and so we find

$$\boxed{G\,|{}^{(\pm)}\Psi_1^g\rangle = g\,|{}^{(\pm)}\Psi_1^g\rangle, \qquad C\,|{}^{(\pm)}\Psi_1^g\rangle = g\,|{}^{(\mp)}\Psi_1^g\rangle,} \tag{48}$$

Therefore, a charged $K\bar{K}$ system must have $g = (-)^{\ell+1}$. Note the subtle difference of the formulas above and (45) for $Q = 0$.

[d] Note that an extra factor of $1/\sqrt{2}$ has been inserted in front of the $K_S\,K_L$ final state in (46d), in order to preserve the normalization. This can be traced to that fact that all the final states in the eqation are orthogonal. It should be emphasized that both (46d) and (46e) are therefore consistent.

6 Three-Particle State: A $K\bar{K}\pi$ System

As an example of the treatment required for three-particle states, we shall work out the case of a neutral $K\bar{K}\pi$ system, which may be in an isospin state of either 0 or 1. Independent of the I-spin, we will show that a wave function for the neutral $K\bar{K}\pi$ system can be devised such that it is in an eigenstate of C with its eigenvalue ± 1 for the state $(K\bar{K})^0\pi^0$ but that it is in an eigenstate of G with its eigenvalue of ± 1 for the state $(K\bar{K})^{\pm}\pi^{\mp}$. In addition, we will also work out in detail the wave functions which result in the case of an intermediate state $K^*(890)$ or $K^*(1420)$. For brevity of notation, we forego the use of ket states used in previous sections; whenever possible, we use the particle names themselves to represent their wave functions.

Consider an arbitrary amplitude A for the state $K\bar{K}\pi$ and expand it in terms of the orbital angular-momentum states for the $K\bar{K}$ subsystem:

$$A = \sum_{\ell} a_{\ell}\psi_{\ell}(K\bar{K}) \tag{49}$$

where the argument denotes ordering of particle momenta in the sense that K has $+\vec{k}$ and $\bar{K}$ has $-\vec{k}$ in the $K\bar{K}$ CM system. Denote by B the amplitude resulting from A by interchanging K and $\bar{K}$. Then, we find

$$\begin{aligned} B &= \sum_{\ell} a_{\ell}\psi_{\ell}(\bar{K}K) \\ &= \sum_{\ell}(-)^{\ell} a_{\ell}\psi_{\ell}(K\bar{K}) \end{aligned} \tag{50}$$

where the second line derives from the property of the spherical harmonics [see(28)]. We can now construct two orthogonal wave functions:

$$\Psi_+ = \frac{1}{2}(A+B) = \sum_{\ell=\text{even}} a_{\ell}\psi_{\ell}(K\bar{K}) \tag{51}$$

$$\Psi_- = \frac{1}{2}(A-B) = \sum_{\ell=\text{odd}} a_{\ell}\psi_{\ell}(K\bar{K}) \tag{52}$$

A neutral $K\bar{K}\pi$ system comes in two varieties: $(K\bar{K})^0\pi^0$ and $(K\bar{K})^{\pm}\pi^{\mp}$. From the previous section, we know that a $(K\bar{K})^0$ system has the C-parity $(-)^{\ell}$ and a π^0 has the positive C-parity, so that

$$\boxed{C\Psi_\pm = \pm\Psi_\pm \qquad \text{for} \quad (K\bar{K})^0\pi^0} \tag{53}$$

This shows that a neutral $K\bar{K}\pi$ state with a π^0 has the C-parity $+1$ if its wave function is even under K and $\bar{K}$ interchange, and the C-parity is -1 if it is odd under the interchange. A charged $K\bar{K}$ system has necessarily $I = 1$. Therefore, the G-parity for $(K\bar{K})^\pm$ is $(-)^{\ell+1}$. But then π has an odd G- parity, so that

$$\boxed{G\Psi_\pm = \pm\Psi_\pm \quad \text{for} \quad (K\bar{K})^\pm\pi^\mp} \tag{54}$$

In other words, a neutral $K\bar{K}\pi$ state with a charged π has the G-parity $+1(-1)$ if the wave function is even(odd) under the K and $\bar{K}$ interchange.

Let us recapitulate the results so far for a neutral $K\bar{K}\pi$ system in a given eigenstate of I, C and G. Regardless of the I-spin for the three-body system, its C-parity is even(odd) if the $K\bar{K}$ pair has even(odd) orbital angular momenta for the state $(K\bar{K})^0\pi^0$, whereas its G-parity is even(odd) if the $K\bar{K}$ pair has even(odd) angular momenta for the state $(K\bar{K})^\pm\pi^\mp$. It is worth emphasizing that, although π^0 is also a G-eigenstate, the G-parity for $(K\bar{K})^0\pi^0$ is not known until the I-spin for the three-body state is determined, since a neutral $K\bar{K}$ system can come with its isospin either 0 or 1. Note that these results hold whatever sequential decay the $K\bar{K}\pi$ system may undergo. In particular, a $K\pi$ intermediate state can be either $J^{PC} = 1^{--}$ $K^*(890)$ or $J^{PC} = 2^{++}$ $K^*(1420)$. Therefore, C or G eigenstates depend on the properties of K and $\bar{K}$ interchange and *not* on the intrinsic parities of the nonets to which $K\pi$ or $\bar{K}\pi$ may belong.

We now turn to the problem of constructing a complete wave function for $K\bar{K}\pi$ with an intermediate $I = 1/2$ K^*. We first start by writing down the K^* states with strangeness$= \pm 1$:

$$\begin{cases} K^{*+} = \sqrt{\frac{2}{3}}\,\pi^+K^0 - \sqrt{\frac{1}{3}}\,\pi^0K^+ \\ K^{*0} = \sqrt{\frac{1}{3}}\,\pi^0K^0 - \sqrt{\frac{2}{3}}\,\pi^-K^+ \end{cases} \qquad \begin{cases} \bar{K}^{*0} = \sqrt{\frac{2}{3}}\,\pi^+K^- - \sqrt{\frac{1}{3}}\,\pi^0\bar{K}^0 \\ K^{*-} = \sqrt{\frac{1}{3}}\,\pi^0K^- - \sqrt{\frac{2}{3}}\,\pi^-\bar{K}^0 \end{cases} \tag{55}$$

A neutral $K^*\bar{K}$ system in an isospin I can be written

$$A_I = \sqrt{\frac{1}{2}}[K^{*+}K^- - (-)^I K^{*0}\bar{K}^0] \tag{56}$$

where I=0 or 1. Combining (55) and (56), we obtain

$$A_I = \sqrt{\frac{1}{3}}[(\pi^+K^0)K^- + (-)^I(\pi^-K^+)\bar{K}^0] \\ -\sqrt{\frac{1}{6}}[(\pi^0K^+)K^- + (-)^I(\pi^0K^0)\bar{K}^0] \tag{57}$$

where parentheses are used to denote K^* states and the ordering of particle names indicates different momenta, e.g. $\vec{p}_1, \vec{p}_2$ and $\vec{p}_3$ in the three-body CM system. Let us denote by B_I the wave function which is obtained from A_I by K and $\bar{K}$ interchange.

We can then define

$$\Phi_g = \sqrt{\frac{1}{2}}[A_I + gB_I] = \Gamma_g + \Theta_g \tag{58}$$

where $g = \pm 1$. Collecting $\pi^{\pm}$'s into Γ_g and π^0's into Θ_g, one finds

$$\begin{aligned} \Gamma_g &= \sqrt{\frac{1}{6}}\Big\{\big[(\pi^+K^0)K^- + g(\pi^+K^-)K^0\big] \\ &\qquad + (-)^I\big[(\pi^-K^+)\bar{K}^0 + g(\pi^-\bar{K}^0)K^+\big]\Big\} \end{aligned} \tag{59}$$

$$\begin{aligned} \Theta_g &= -\sqrt{\frac{1}{12}}\Big\{\big[(\pi^0K^+)K^- + g\,(\pi^0K^-)K^+\big] \\ &\qquad + (-)^I\,\big[(\pi^0K^0)\bar{K}^0 + g\,(\pi^0\bar{K}^0)K^0\big]\Big\} \end{aligned} \tag{60}$$

In order to gain insight to the above formula, it is helpful to define $K\bar{K}$ amd $\bar{K}K$ states in isospin σ,

$$F_\sigma = \sqrt{\frac{1}{2}}[K^+K^- - (-)^\sigma K^0\bar{K}^0] \tag{61}$$

$$\bar{F}_\sigma = \sqrt{\frac{1}{2}}[\bar{K}^0K^0 - (-)^\sigma K^-K^+] \tag{62}$$

These formulae show that neutral $K\bar{K}$ states of (60) have isospins $\sigma = 1$ if $I = 0$ and $\sigma = 0$ if $I = 1$. We conclude, therefore, under the G operation,

$$G\,\Gamma_g = g\,\Gamma_g \tag{63}$$

$$G\,\Theta_g = g\,(-)^I\Theta_g \tag{64}$$

$$C\,\Theta_g = g\,\Theta_g \tag{65}$$

We find once more that a neutral $K\bar{K}\pi$ system with a charged π has a G-parity g, whereas its G-parity is $g(-)^I$ if it has a π^0. In other words, it is the C-parity which is independent of I for the $(K\bar{K})^0\pi^0$ system. From (63) and (64), it is clear that g takes on a different meaning depending on the functions; this is a consequence of the fact that one is exploring the properties of a wave function symmetrized through interchange of K and $\bar{K}$. One may, instead, construct an eigenstate of G-parity alone, which leads to a wave function different from (58). For a thorough analysis of this and other related topics, the reader is referred to another note[1].

It is frequently the case in experiments that both $(K\bar{K})^+\pi^-$ and $(K\bar{K})^-\pi^+$ are measured. Each event of course comes in either one or the other charge state, so that the two charge states cannot interfere with each other. Assuming that the events are dominated either by $I = 0$ states or by $I = 1$ states but not both[1], one may combine the two data sets in two different ways. Since the three variables describing orientation of the three-particle system can be defined equally for the two data sample, one needs to investigate only the two Dalitz-plot variables. For the purpose,let us define the squares of $K\pi$ effective masses,

$$s_1 = M^2(\pi^+K^0), \qquad s_2 = M^2(\pi^+K^-) \tag{66}$$
$$s_3 = M^2(\pi^-K^+), \qquad s_4 = M^2(\pi^-\bar{K}^0)$$

Now we need to write down the general amplitude for the two data sample separately, from (59),

$$\phi_g^+ = (\pi^+K^0)K^- + g(\pi^+K^-)K^0 \tag{67}$$
$$\phi_g^- = (\pi^-K^+)\bar{K}^0 + g(\pi^-\bar{K}^0)K^+ \tag{68}$$

where the superscripts designate π^+ and π^- charge states. We can assume, quite generally, that both charge states produce the same admixture of G-parity eigenstates:

$$\phi^+ = a\phi_+^+ + b\phi_-^+ \tag{69}$$
$$\phi^- = a\phi_+^- + b\phi_-^- \tag{70}$$

where a and b are two arbitrary complex numbers.

In combining the two data, it is natural to respect the strangeness and equate the Dalitz-plot variables as $s_1 = s_3$ and $s_2 = s_4$. An examination of (67) and (68) shows that this amounts to equating the amplitudes as follows:

$$D = (\pi^+ K^0)K^- = (\pi^- K^+)\bar{K}^0 \tag{71}$$
$$E = (\pi^+ K^-)K^0 = (\pi^- \bar{K}^0)K^+$$

Submitting these into (69) and (70), one finds

$$\phi^+ = a(D + E) + b(D - E) \tag{72}$$

$$\phi^- = a(D + E) + b(D - E) \tag{73}$$

One has an option of combining $K\pi$ variables according to their net charge i.e. neutral or charged. In that case, one then sets $s_1 = s_4$ and $s_2 = s_3$. The amplitudes are now to be combined in the following way:

$$D = (\pi^+ K^0)K^- = (\pi^- \bar{K}^0)K^+ \tag{74}$$
$$E = (\pi^+ K^-)K^0 = (\pi^- K^+)\bar{K}^0$$

These lead to the overall amplitudes

$$\phi^+ = a(D + E) + b(D - E) \tag{75}$$

$$\phi^- = a(D + E) - b(D - E) \tag{76}$$

The difference between the two approaches is now apparent; it the minus sign in the second term in (76) which is different from (73). Note that, when one combines two charge samples, those with a π^+ and with a π^-, one is in fact combining the squares of the amplitudes

$$|\phi|^2 = |\phi^+|^2 + |\phi^-|^2 \tag{77}$$

This shows that, in the latter approach in which neutral and charged $K\pi$ variables are used, one is cancelling out the G-parity plus-and-minus interference terms (the terms with a*b). On the other hand, in the former approach in which the $K\pi$ variables are grouped together according their strangeness, one is in fact reinforcing the interference term. The reason for this difference can be traced to the fact that the strangeness is a conserved quantum number in strong interactions. It should be emphasized that, in a spin-parity analysis, presence of two G-parity states is detected most sensitively through their interference term.

We end this section by commenting on the branching ratios one expects with a K^* intermediate state. For the purpose, it is best to go back to (57). We see that $BR = 1/3$ for both $(\pi^+ K^0)K^-$ and $(\pi^- K^+)\bar{K}^0$ whereas $BR = 1/6$ for $(\pi^0 K^+)K^-$ and $(\pi^0 K^0)\bar{K}^0$. If neutral K states are detected through $K_S \to \pi^+\pi^-$, one needs to augment the branching ratios by an additional factor $\approx 1/3$ for each K^0 and/or $\bar{K}^0$ appearing in each $K\bar{K}\pi$ state. Note that these branching ratios are independent of the total I-spin for the three-body system.

For a more complete treatment of the $K\bar{K}\pi$ systems, the reader is referred to the notes by the author[1], [2].

7 Four-Particle State: A $K\bar{K}\pi\pi$ System

If the $K\bar{K}$ system forms an isobar, then its G-parity is in fact equal to that of the four-body system considered here, because the $\pi\pi$ system is already in a G-parity eigenstate with the eigenvalue $+1$. The G-parity eigenstates of $K\bar{K}$ systems has already been treated in Section 5. We therefore only need to work out G-parity eigenstates of $K^*\,\bar{K}^*$+ c.c. We may write, with $\sigma_1 = \sigma_2 = 1/2$,

$$\mathcal{A}^g(I\nu) = \frac{1}{\sqrt{2}} \sum_{\nu_1\,\nu_2} (\sigma_1\nu_1\;\sigma_2\nu_2|I\nu)\;\Big[(K^*\,\sigma_1\nu_1)\;(\bar{K}^*\,\sigma_2\nu_2) - g\,(\bar{K}^*\,\sigma_1\nu_1)\;(K^*\,\sigma_2\nu_2)\Big] \tag{78}$$

We see that, from (22),

$$\boxed{G\;\mathcal{A}^g(I\nu) = g\,\mathcal{A}^g(I\nu)} \tag{79}$$

For ease of reference, we work out explicitly for $I = 0$ or 1

$$\begin{aligned}\mathcal{A}^g(I0) = \frac{1}{2}\Big[&(K^*\,\sigma_1\;+1/2)\;(\bar{K}^*\,\sigma_2\;-1/2) - (-)^I(K^*\,\sigma_1\;-1/2)\;(\bar{K}^*\,\sigma_2\;+1/2)\\ &- g\,(\bar{K}^*\,\sigma_1\;+1/2)\;(K^*\,\sigma_2\;-1/2) + g(-)^I\,(\bar{K}^*\,\sigma_1\;-1/2)\;(K^*\,\sigma_2\;+1/2)\Big]\end{aligned} \tag{80}$$

and for $I = 1$,

$$\mathcal{A}^g(1\pm1) = \frac{1}{\sqrt{2}}\Big[(K^*\,\sigma_1\;\pm1/2)\;(\bar{K}^*\,\sigma_2\;\pm1/2) - g\,(\bar{K}^*\,\sigma_1\;\pm1/2)\;(K^*\,\sigma_2\;\pm1/2)\Big] \tag{81}$$

In a reaction with diffractive dissociation of a negative-pion beam (the Pomeron exchange reaction), the produced system preserves the flavor of a negative pion; so it must have $I = 1$, $Q = -1$ and $g = -1$. It is helpful to write down the formula (81) again but with K^*'s given in (55)

$$\begin{cases} \mathcal{A}^g(1\ +1) = \dfrac{1}{\sqrt{2}}\Big[K^{*+}\ \bar{K}^{*0} - g\,\bar{K}^{*0}\ K^{*+}\Big] \\ \qquad = \dfrac{1}{\sqrt{2}}\Bigg[\left(\sqrt{\dfrac{2}{3}}\,\pi^+K^0 - \sqrt{\dfrac{1}{3}}\,\pi^0K^+\right)\left(\sqrt{\dfrac{2}{3}}\,\pi^+K^- - \sqrt{\dfrac{1}{3}}\,\pi^0\bar{K}^0\right) \\ \qquad\quad - g\left(\sqrt{\dfrac{2}{3}}\,\pi^+K^- - \sqrt{\dfrac{1}{3}}\,\pi^0\bar{K}^0\right)\left(\sqrt{\dfrac{2}{3}}\,\pi^+K^0 - \sqrt{\dfrac{1}{3}}\,\pi^0K^+\right)\Bigg] \\ \mathcal{A}^g(1\ -1) = \dfrac{1}{\sqrt{2}}\Big[K^{*0}\ K^{*-} - g\,K^{*-}\ K^{*0}\Big] \\ \qquad = \dfrac{1}{\sqrt{2}}\Bigg[\left(\sqrt{\dfrac{1}{3}}\,\pi^0K^0 - \sqrt{\dfrac{2}{3}}\,\pi^-K^+\right)\left(\sqrt{\dfrac{1}{3}}\,\pi^0K^- - \sqrt{\dfrac{2}{3}}\,\pi^-\bar{K}^0\right) \\ \qquad\quad - g\left(\sqrt{\dfrac{1}{3}}\,\pi^0K^- - \sqrt{\dfrac{2}{3}}\,\pi^-\bar{K}^0\right)\left(\sqrt{\dfrac{1}{3}}\,\pi^0K^0 - \sqrt{\dfrac{2}{3}}\,\pi^-K^+\right)\Bigg] \end{cases} \tag{82}$$

Collecting the terms with $\pi^\pm$ (i.e. dropping those with π^0's), we obtain, after renormalizing,

$$\begin{cases} \mathcal{L}^g(1\ +1) = \dfrac{1}{\sqrt{2}}\Big[(\pi^+K^0)(\pi^+K^-) - g\,(\pi^+K^-)(\pi^+K^0)\Big] \\ \mathcal{L}^g(1\ -1) = \dfrac{1}{\sqrt{2}}\Big[(\pi^-K^+)(\pi^-\bar{K}^0) - g\,(\pi^-\bar{K}^0)(\pi^-K^+)\Big] \end{cases} \tag{83}$$

where the parentheses indicate the particles which form K^*'s. The equation (83) shows that the wave function with $Q = \pm 1$ appears completely symmetrized for $g = -1$; this would be the case if the $(K\bar{K}\pi\pi)^\pm$ systems were produced via diffractive dissociation from $\pi^\pm$ beams.

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Appendix

We start with K_1 and K_2 defined in (23)

$$\begin{cases} |K_1\rangle = \dfrac{1}{\sqrt{2}}\Big[|K^0\rangle + |\bar{K}^0\rangle\Big] \\ |K_2\rangle = \dfrac{1}{\sqrt{2}}\Big[|K^0\rangle - |\bar{K}^0\rangle\Big] \end{cases} \qquad \begin{cases} |K^0\rangle = \dfrac{1}{\sqrt{2}}\Big[|K_1\rangle + |K_2\rangle\Big] \\ |\bar{K}^0\rangle = \dfrac{1}{\sqrt{2}}\Big[|K_1\rangle - |K_2\rangle\Big] \end{cases} \tag{84}$$

We note that, from (22),

$$|\bar{K}^0\rangle = CP\,|K^0\rangle \qquad \text{and} \qquad |K^0\rangle = CP\,|\bar{K}^0\rangle \tag{85}$$

and so

$$\boxed{CP\,|K_1\rangle = +|K_1\rangle \qquad \text{and} \qquad CP\,|K_2\rangle = -|K_2\rangle} \tag{86}$$

The neutral K^0 and $\bar{K}^0$ can be recast into K_S and K_L, expressed in terms of the CP-violating complex parameters ϵ_S and ϵ_L, with $\epsilon_S = \epsilon + \delta$ and $\epsilon_L = \epsilon - \delta$,

$$\begin{cases} |K_S\rangle = \dfrac{1}{\sqrt{1+|\epsilon_S|^2}}\Big[|K_1\rangle + \epsilon_S\,|K_2\rangle\Big] \\ |K_L\rangle = \dfrac{1}{\sqrt{1+|\epsilon_L|^2}}\Big[|K_2\rangle + \epsilon_L\,|K_1\rangle\Big] \end{cases} \tag{87}$$

The parameters ϵ_S and ϵ_L are small (see below), and so the K_S and K_L are approximately equal to K_1 and K_2, which are CP-even and CP-odd eigenstates, respectively.

The K_S and K_L are mass eigenstates, whose masses are nearly equal, with or without the CPT invariance:

$$m(K_L) - m(K_S) \simeq (3.483 \pm 0.006) \times 10^{-12}\ \text{MeV}, \quad (\text{assuming } CPT) \tag{88}$$

The mass difference is nearly the same with CPT but its error is nearly twice that quoted above. The two neutral K mesons are distinguished by their mean life times and their major decay modes, i.e.

$$c\tau(K_S) = 2.6842\ \text{cm} \quad (\text{assuming } CPT), \qquad c\tau(K_L) = 15.34\ \text{m} \tag{89}$$

and

$$\begin{cases} K_S \to (\pi\pi)^0 & \text{BR} \simeq 99.89\% \\ K_L \to (3\pi)^0 & \text{BR} \simeq 32.06\% \end{cases} \qquad \begin{cases} K_L \to (\pi\, e)^0 \nu_e & \text{BR} \simeq 40.55\% \\ K_L \to (\pi\, \mu)^0 \nu_\mu & \text{BR} \simeq 27.04\% \end{cases} \tag{90}$$

Now if we substitute (84) into (87), we obtain [e]

$$\begin{cases} |K_S\rangle = \dfrac{1}{\sqrt{2(1+|\epsilon_S|^2)}} \Big[(1+\epsilon_S)\, |K^0\rangle + (1-\epsilon_S)\, |\bar{K}^0\rangle \Big] \\ |K_L\rangle = \dfrac{1}{\sqrt{2(1+|\epsilon_L|^2)}} \Big[(1+\epsilon_L)\, |K^0\rangle - (1-\epsilon_L)\, |\bar{K}^0\rangle \Big] \end{cases} \tag{91}$$

and its inverse is

$$\begin{cases} |K^0\rangle = \dfrac{1}{\sqrt{2}\,(1-\epsilon_S\epsilon_L)} \Big[(1-\epsilon_L)\sqrt{1+|\epsilon_S|^2}\, |K_S\rangle + (1-\epsilon_S)\sqrt{1+|\epsilon_L|^2}\, |\bar{K}_L\rangle \Big] \\ |\bar{K}^0\rangle = \dfrac{1}{\sqrt{2}\,(1-\epsilon_S\epsilon_L)} \Big[(1+\epsilon_L)\sqrt{1+|\epsilon_S|^2}\, |K_S\rangle - (1+\epsilon_S)\sqrt{1+|\epsilon_L|^2}\, |\bar{K}_L\rangle \Big] \end{cases} \tag{92}$$

According to the PDG Book (July 2010), we have

$$\begin{aligned} \epsilon_S &= \epsilon + \delta \quad \text{and} \quad \epsilon_L = \epsilon - \delta \\ |\epsilon| &= (2.228 \pm 0.011) \times 10^{-3} \text{ (PDG, page 38)} \\ \Re(\epsilon) &= (1.612 \pm 0.006) \times 10^{-3} \text{ (PDG, page 760)} \\ \Im(\epsilon) &= (1.539 \pm 0.010) \times 10^{-3} \text{ (derived from the two numbers above)} \\ \Re(\delta) &= (0.25 \pm 0.23) \times 10^{-3} \text{ (PDG, page 760)} \\ \Im(\delta) &= (-0.006 \pm 0.019) \times 10^{-3} \text{ (PDG, page 760)} \end{aligned} \tag{93}$$

In the limit of the CPT invariance, we must have $\delta = 0$, so that $\epsilon_S = \epsilon_L$. In this limit we observe that (92) simplifies to

$$\begin{cases} |K^0\rangle = \dfrac{\sqrt{1+|\epsilon|^2}}{\sqrt{2}\,(1+\epsilon)} \Big[|K_S\rangle + |\bar{K}_L\rangle \Big] \\ |\bar{K}^0\rangle = \dfrac{\sqrt{1+|\epsilon|^2}}{\sqrt{2}\,(1-\epsilon)} \Big[|K_S\rangle - |\bar{K}_L\rangle \Big] \end{cases} \tag{94}$$

[e] There is a sign error in the PDG (July 2010), page 759. The second term of K_L must have a minus sign (not a plus sign). The same error is present in the previous PDG (July 2008), page 722.

References

[1] S. U. Chung, 'Analysis of $K\bar{K}\pi$ systems (Version II),' BNL-QGS-98-901

[2] S. U. Chung, 'Formulas for Partial-Wave Analysis (Version V),' BNL-QGS-06-102