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Central Production of Two- and Four-pion Systems at COMPASS and ALICE

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Abstract

The amplitude analysis of two- and four-pseudoscalar systems is presented for the central production at COMPASS and ALICE. The production is mediated by exchanges of $\boldsymbol{P}+\boldsymbol{P}$ and $\boldsymbol{P}$+Reggeon. Two- to three-body processes require two planes in which to imbed the relevant momenta in the rest frame of the centrally produced state. Parity conservation for the central production is worked out using the reflectivity operators defined in the two planes.

WORK IN PROGRESS

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1 Introduction

This note is an extended version of a paper[1] on the ambiguous solutions of the $\pi\pi$ systems by the author, adapted to the central production of $\pi\pi$ and $\pi\pi\pi\pi$ for X^0 in

$$a + b \to 1 + 3(X^0) + 2; \quad a \to 1 + c(\boldsymbol{P}); \quad b \to 2 + d(\boldsymbol{P}) \tag{1}$$

via double-Pomeron exchange at COMPASS and ALICE, see Fig. 1:

$$c(\boldsymbol{P}) + d(\boldsymbol{P}) \to 3(X^0) \to (\pi\pi)^0 \quad \text{or} \quad (\pi\pi\pi\pi)^0 \tag{2}$$

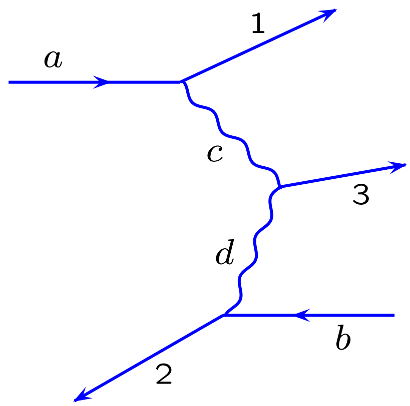

Figure 1: Production of a system 3 from the reaction $a + b \to 1 + 3 + 2$.
Here c and d stand for the exchanged Reggeons (or Pomerons).

For COMPASS, we take the x-z plane to be that formed by the vertex c-d-$3(X^0)$, so that the production normal (the y-axis) is along $\vec{c} \times \vec{d}$ in the X^0 rest frame. We choose the z-axis to be along the Reggeon (Pomeron) $c(\boldsymbol{P})$ such that $|t_c| \leq |t_d|$. So Pomerons are distinguished by their $|t|$; so they are not identical particles. The system X^0 must have the quantum numbers, in the limit of double-Pomeron exchange,

$$\begin{aligned} I^G(J^{PC}) &= 0^+(0^{++}),\ 0^+(2^{++}),\ 0^+(4^{++}),\ 0^+(6^{++}), \cdots \quad \text{for} \quad (\pi\pi)^0 \\ &= 0^+(0^{\pm+}),\ 0^+(1^{\pm+}),\ 0^+(2^{\pm+}),\ 0^+(3^{\pm+}),\ 0^+(4^{\pm+}) \cdots \quad \text{for} \quad (\pi\pi\pi\pi)^0 \end{aligned} \tag{3}$$

For an even-pion system, the G-parity is always positive, and $I + \ell =$ even where ℓ is the spin of the $(\pi\pi)^0$ system. If the production proceeds via a double-Pomeron exchange, then an isovector cannot be produced. The isovector states are possible if one of the exchanged Reggeon is the $\rho(770)$. We note here that, for a Pomeron-Reggeon exchange, the exchanged Reggeon cannot be an $\omega(782)$, since we deal with a system with $G = +1$.

For ALICE, we take the x-z plane to be that formed by $\vec{a}$ and $\vec{b}$. Then the y-axis is proportional to $\vec{a} \times \vec{b}$, again in the X^0RF (rest frame). The system X^0 must have, in the limit of double-Pomeron exchange,

$$\begin{aligned} I^G(J^{PC}) &= 0^+(0^{++}),\ 0^+(2^{++}),\ 0^+(4^{++}),\ 0^+(6^{++}),\cdots \quad \text{for} \quad (\pi\pi)^0 \\ &= 0^+(0^{\pm+}),\ 0^+(1^{\pm+}),\ 0^+(2^{\pm+}),\ 0^+(3^{\pm+}),\ 0^+(4^{\pm+})\cdots \quad \text{for} \quad (\pi\pi\pi\pi)^0 \end{aligned} \tag{4}$$

because of the Bose symmetry for two $J^{PC} = 2^{++}$ Pomerons ($S + j$ = even where $S = 0, 1, 2, 3, 4$)[2] and because of the fact that $I + j$ = even for a $\pi\pi$ system. Note that the produced dipion system is always an isoscalar with even G-parity, i.e. C is always positive.

This note is intended for beginning graduate students; as such it contains much more detailed intermediate steps in a number of derivations worked out, clearly more than customary and/or appropriate for publication in physics journals.

For interested readers, we have collected all the references cited in this note, as well as this note itself in

`http://cern.ch/suchung/`

a public website at CERN.

For a comprehensive review, we would suggest two recent books, "Pomeron Physics and QCD," by S. Donnachie, G. Dosch, P. Landshoff and O. Nachtmann[3] and "High-Energy Particle Diffraction," by V. Barone and E. Predazzi[4].

2 Reflection Operators

We shall examine the eigenstates of the reflection operator, with η and p standing for the intrinsic parity and four-momentum p of X^0, with the requirement that $\vec{p}$ lies in the x-z plane,

$$|p, \epsilon jm\rangle = \theta(m)\Big\{|p, jm\rangle + \eta\,\epsilon(-)^{j-m}|p, j\ \ -m\rangle\Big\} \tag{5}$$

where

$$\begin{aligned} \theta(m) &= \frac{1}{\sqrt{2}}, \quad m > 0 \\ &= \frac{1}{2}, \quad m = 0 \\ &= 0, \quad m < 0 \end{aligned} \tag{6}$$

One sees that $\tau(m) = 4\theta^2(m)$, see (48). We explore the actions of the reflection operator

$$\Pi_y = \Pi\, R_y(\pi) = R_y(\pi)\, \Pi \tag{7}$$

for the reaction (1) and (2), where $R_y(\pi)$ represents the rotation by π around the y-axis for a reaction which takes place in the x-z plane. We use the relationaship

$$\langle j\, m'|R_y(\pi)|j\, m\rangle = d^j_{m'\,m}(\pi) = (-)^{j-m}\,\delta_{m'\,-m} \tag{8}$$

and work out the transformation of *a rest-frame ket state* under Π_y

$$\begin{aligned}\Pi_y\,|j\,m\rangle &= \eta\,R_y(\pi)|j\,m\rangle = \eta\sum_{m'}|j\,m'\,\rangle\langle j\,m'\,|R_y(\pi)|j\,m\rangle \\ &= \eta\sum_{m'}|j\,m'\,\rangle\,d^j_{m'\,m}(\pi) = \eta\,(-)^{j-m}\,|j\;\;-m\rangle\end{aligned} \tag{9}$$

and find, since $\eta^2 = +1$,

$$\Pi_y^2\,|j\,m\rangle = (-)^{2j}\,|j\,m\rangle \tag{10}$$

a well-known formula for a state undergoing a roation of 2π. A *relativitistic* ket state, including that in the helicity basis, transforms according to

$$\Pi_y|p,jm\rangle = \eta(-)^{j-m}\,|p,j\;\;-m\rangle \tag{11}$$

provided that $\vec{p}$ lies in the x-z plane, so that the the Lorentz transformation needed to go from a rest state to a *relativitistic* ket state with four-momentum p commutes with the reflection operator Π_y, since the y-axis is normal to the direction of the Lorentz transformation. So we find

$$\Pi_y\,|p,\epsilon jm\rangle = \epsilon(-)^{2j}\,|p,\epsilon jm\rangle \tag{12a}$$
$$\Pi_y^2\,|p,\epsilon jm\rangle = \epsilon^2(-)^{4j}\,|p,\epsilon jm\rangle = \epsilon^2\,|p,\epsilon jm\rangle(12b) \tag{12b}$$
$$\equiv (-)^{2j}\,|p,\epsilon jm\rangle \tag{12c}$$

The formula (12c) comes from the requirement that the states $|p,\epsilon jm\rangle$ statisfy (10).[a] So we see that $\epsilon^2 = (-)^{2j}$, or equivalently $\epsilon^2(-)^{2j} = +1$. We note that $(-)^{4j} = +1$, since $4j$ an even integer always. It follows that $\epsilon = \pm 1$ for bosons and $\epsilon = \pm i$ for fermions. Note, in addition, that $\epsilon\epsilon^* = +1$ in general, and so $\epsilon = \epsilon^*(-)^{2j}$. Note that $(\epsilon^*)^2 = (-)^{-2j}$. We shall later use the bra form of (5)

$$\langle p,\epsilon jm| = \theta(m)\Big\{\langle p,jm| + \eta\,\epsilon^*\,(-)^{j-m}\langle p,j\;\;-m|\Big\} \tag{13}$$

We now write down the particles of two- to three-body reaction for which all the particles are given in helicity-basis vectors except for the system X^0 which will be given in the reflectivity basis shown in (5). The transition amplitude can be written

$${}^{\epsilon}V_{j\,m\,k} = \langle q_1, j_1\lambda_1; q_3, \epsilon j\,m; q_2, j_2\lambda_2|\mathcal{M}|q_a, j_a\lambda_a; q_b, j_b\lambda_b\rangle \tag{14a}$$
$$(13) \to = \theta(m)\Big\{V_{j\,m\,k} + \eta\,\epsilon^*\,(-)^{j-m}\,V_{j\,-m\,k}\Big\} \tag{14b}$$

where $\mathcal{M}$ is the operator representing the transition amplitude of reaction (1) and q_i stands for a four-momentum in the overall center-of-mass frame for COMPASS and the laboratory frame for ALICE. Here the index k stand for all the helicities appearing on the right-hand

[a] This can derived most expeditiously by applying Π_y^2 separately to the first term of (5) first and the second term next. It is clear, from the method outlined here, that the two terms revert back to their original forms and that the result cannot depend on ϵ, except that there is merely a common additional factor $(-)^{2j}$ as in (10).

side of (14a), i.e. the helcities for particles a, b, 1 and 2. Assuming factorization holds for the reaction (1), we can write

$$
\begin{aligned}
{}^{\epsilon}V_{j\,m\,k} = \sum_{j_c\lambda_c;\,j_d\lambda_d} & \langle q_3, \epsilon j\, m|\mathcal{M}_3|q_c, j_c\lambda_c; q_d, j_d\lambda_d\rangle \\
& * \langle q_1, j_1\lambda_1; q_c, j_c\lambda_c|\mathcal{M}_a|q_a, j_a\lambda_a\rangle * \langle q_2, j_2\lambda_2; q_d, j_d\lambda_d|\mathcal{M}_b|q_b, j_b\lambda_b\rangle
\end{aligned}
\tag{15}
$$

where the subscripts a, b and 3 of $\mathcal{M}$ indicate the appropriate transition operators for the initial states a, b and the time-reversed reaction $X^0 \to cd$. We first go into the rest frame of X^0 (or, equivalently, the 3RF, see Fig. 1). We use the symbols $\vec{q}$ for the 3-momenta and $\mathcal{E}$ for the energy in the X^0RF. We see that, since $\vec{q}_3 = 0$,

$$
\begin{aligned}
&\vec{q}_a + \vec{q}_b = \vec{q}_1 + \vec{q}_2 \equiv \vec{q}_e\,, \qquad \mathcal{E}_a + \mathcal{E}_b = \mathcal{E}_1 + \mathcal{E}_2 + m_3 \equiv \mathcal{E}_e \\
&\vec{q}_a = \vec{q}_1 + \vec{q}_c\,, \qquad \vec{q}_b = \vec{q}_2 + \vec{q}_d\,, \qquad \vec{q}_c + \vec{q}_d = 0
\end{aligned}
\tag{16}
$$

Consider now two planes of interest for a stusy of central productions. From (16), we see that, by multiplying the first of the second equation above by $(\times\vec{q}_1)$ from the right and by multiplying the second of the second equation above by $(\times\vec{q}_2)$ from the right,

$$
\vec{q}_a \times \vec{q}_1 = \vec{q}_c \times \vec{q}_1 \qquad \vec{q}_b \times \vec{q}_2 = \vec{q}_d \times \vec{q}_2 \tag{17}
$$

This shows that $\vec{q}_c$ lies in the plane $(a|1c)$ contains, while $\vec{q}_d$ lies in another plane $(b|2d)$. But, in the X^0RF, the vectors $\vec{q}_c$ and $\vec{q}_d$ are equal and opposite from each other. This shows that the two planes $(a|1c)$ and $(b|2d)$ intersect along the lines $+\vec{q}_c$ and $-\vec{q}_d$ with the $3(X^0)$ situated in the middle as shown in Fig. 2; the line is in fact along the lines defining the two four-momentum transfers t_c and t_d in the problem[10].

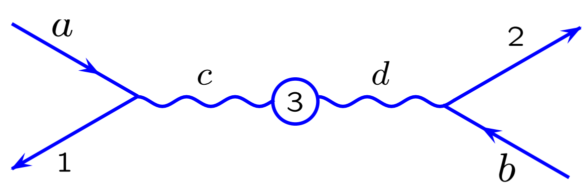

Figure 2: Production of a system 3 via $c + d \to 3$ in the 3 rest frame, where c and d stand for the Reggeons (or Pomerons).

The relevant four-momentum transfers are

$$
\begin{aligned}
t_c &= \mathcal{E}_c{}^2 - q_c^2 & t_d &= \mathcal{E}_c{}^2 - q_c^2 \\
&= (\mathcal{E}_a - \mathcal{E}_1)^2 - (\vec{q}_a - \vec{q}_1)^2 & &= (\mathcal{E}_b - \mathcal{E}_2)^2 - (\vec{q}_b - \vec{q}_2)^2 \\
&= m_a^2 + m_1^2 - 2(\mathcal{E}_a\,\mathcal{E}_1 - \vec{q}_a \cdot \vec{q}_1) & &= m_b^2 + m_2^2 - 2(\mathcal{E}_b\,\mathcal{E}_2 - \vec{q}_b \cdot \vec{q}_2)
\end{aligned}
\tag{18}
$$

We set the z-axis to be along $\vec{q}_c$ in the X^0RF and put the plane $(a|1c)$ to be on the x-z plane, i.e.

$$
\hat{x}_a = (1,0,0) \quad \hat{y}_a = (0,1,0) \quad \hat{z}_a = (0,0,1) \tag{19}
$$

Then the plane $(b|2d)$ is obtained by rotating the $(a|1c)$ plane by Φ, the angle between the planes $(a|1c)$ and $(b|2d)$. Then the rotated coordinate system is given by

$$\hat{x}_b = (\sin\Phi, \cos\Phi, 0, 0) \quad \hat{y}_b = (\cos\Phi, -\sin\Phi, 0) \quad \hat{z}_b = (0, 0, 1) \tag{20}$$

So, by definition, the normal to the plane $(a|1c)$ is along $\hat{y}_a$, while the normal to the plane $(b|2d)$ is $\hat{y}_b$.

The density matrix is, from (15),

$$\begin{aligned}
{}^{\epsilon\,\epsilon'}\!\rho^{jj'}_{mm'} &= \sum_k {}^{\epsilon}V_{jmk}\;{}^{\epsilon'}V^*_{j'm'k} \\
&= \sum_{\substack{\lambda_1\lambda_2;\lambda_a\,\lambda_b \\ j_c\lambda_c\,j_d\lambda_d;j'_c\lambda'_c\,j'_d\lambda'_d}} \langle q_3, \epsilon j\, m|\mathcal{M}_X|q_c, j_c\lambda_c; q_d, j_d\lambda_d\rangle\langle q_3, \epsilon' j'\, m'|\mathcal{M}_X|q_c, j'_c\lambda'_c; q_d, j'_d\lambda'_d\rangle^* \\
&\qquad \times \langle q_1, j_1\lambda_1; q_c, j_c\lambda_c|\mathcal{M}_a|q_a, j_a\lambda_a\rangle\langle q_1, j_1\lambda_1; q_c, j'_c\lambda'_c|\mathcal{M}_a|q_a, j_a\lambda_a\rangle^* \\
&\qquad \times \langle q_2, j_2\lambda_2; q_d, j_d\lambda_d|\mathcal{M}_b|q_b, j_b\lambda_b\rangle\langle q_2, j_2\lambda_2; q_d, j'_d\lambda'_d|\mathcal{M}_b|q_b, j_b\lambda_b\rangle^*
\end{aligned} \tag{21}$$

The process $(cd|X^0)$ shown on the first line above is colinear for $X^0 \to c + d$, aligned along the z-axis. So the reflection operator for this process can be defined with any y-axis as long as it is confined to the x-z plane. We shall use the reflection operator Π_y for this process and apply $\Pi_y^\dagger\,\Pi_y = I$ to the two vertices on the first line above.[b] We obtain a factor $\epsilon(\epsilon')^*$ and the helicities of the particles c and d change sign. We define separately the reflection operators Π^a_y and Π^b_y for $(b|2d)$, for the planes $(a|1c)$ and $(b|2d)$ by respectively. We apply $(\Pi^a_y)^\dagger\Pi^a_y = I$ to the second line above and $(\Pi^b_y)^\dagger\Pi^b_y = I$ to the third line above and work out the necessary transformations of the states involved. The net results is that no multiplicative factors are needed, as the particles a, b, 1 and 2, and c, c', d and d' occur twice in the equations above, but that all the helicities involved change sign. Since the helicities are summed over, the sign changes are immaterial for (21).

As a consequence, the density matrix must satisfy

$${}^{\epsilon\,\epsilon'}\!\rho^{jj'}_{mm'} = \epsilon(\epsilon')^* \times {}^{\epsilon\,\epsilon'}\!\rho^{jj'}_{mm'} \tag{22}$$

so we must have $\epsilon(\epsilon')^* = +1$. Or we obtain, by multiplying by ϵ' from the right, a simple and elegant result $\epsilon = \epsilon'$. This shows that the density matrix breaks up into block-diagonal forms with $\epsilon = +1(+i)$ and $\epsilon = -1(-i)$; there are no interference terms between $\epsilon = +1(+i)$ and $\epsilon = -1(-i)$. Finally, we need to emphasize that the reflection operators are always defined with the y-axis normal to the reaction plane; so the 3-momenta remain the same through the operations, i.e. they remain invariant. We add, in addition, that there are in fact only two reflection operators in the problem, i.e. Π^a_y and Π^b_y. The third reflection operator Π_y, introduced initially for the process $c + d \to X^0$, can be replaced by either Π^a_y or Π^b_y, as the

[b] The identity operator is inserted next to $\mathcal{M}_X$ and the $\Pi_y^\dagger$ is propagated left and Π_y propagated right; the reflection operators commute with $\mathcal{M}_X$ and the Lorentz boost implied by the nonzero 3-momenta present in the problem, so that the reflection operators act directly on the rest-frame ket states. Note that the Lorentz boost is confined to the x-z plane; it is for this reason we conclude that the reflection operators must commute with the Lorentz boosts.

reaction takes place along the z-axis, which is the line of intersection between the planes $(a1|c)$ and $(b2|d)$. So $\vec{c}$ is along $+\hat{z}$ and $\vec{d}$ is along $-\hat{z}$, or *vice versa*, with the X^0 remaining at the origin of the the coordinate system for the X^0RF.

We are now ready to derive an important formula for the density matrix. For the purpose, we start with (14b)

$$\begin{aligned}
{}^{\epsilon}\rho^{jj'}_{mm'} &= \sum_k {}^{\epsilon}V_{jmk}\, {}^{\epsilon}V^*_{j'm'k} \\
&= \theta(m)\theta(m') \sum_k \left\{ V_{j\,m\,k} + \eta\,\epsilon^*\,(-)^{j-m}\, V_{j\,-m\,k} \right\} \\
&\qquad\qquad\qquad \times \left\{ V^*_{j'\,m'\,k} + \eta'\,\epsilon\,(-)^{j'-m'}\, V^*_{j'\,-m'\,k} \right\}
\end{aligned} \tag{23}$$

and evaluate, for non-negative values of m and m',

$$\begin{aligned}
{}^{\epsilon}\rho^{\,j\,j'}_{-m\,-m'} &= \sum_k {}^{\epsilon}V_{j\,-mk}\, {}^{\epsilon}V^*_{j'\,-m'k} \\
&= \theta(m)\theta(m') \sum_k \left\{ V_{j\,-m\,k} + \eta\,\epsilon^*\,(-)^{j+m}\, V_{j\,m\,k} \right\} \\
&\qquad\qquad\qquad \times \left\{ V^*_{j'\,-m'\,k} + \eta'\,\epsilon\,(-)^{j'+m'}\, V^*_{j'\,m'\,k} \right\}
\end{aligned} \tag{24}$$

Note that

$$\begin{aligned}
\eta\epsilon^*(-)^{j-m} \times \eta\epsilon^*(-)^{j+m} &= (\epsilon^*)^2(-)^{2j} = +1 \\
\eta'\epsilon(-)^{j'-m'} \times \eta'\epsilon(-)^{j'+m'} &= (\epsilon)^2(-)^{2j'} = +1
\end{aligned} \tag{25}$$

so that

$$\begin{aligned}
{}^{\epsilon}\rho^{\,j\,j'}_{-m\,-m'} &= \sum_k {}^{\epsilon}V_{j\,-mk}\, {}^{\epsilon}V^*_{j'\,-m'k} \\
&= \eta\,\epsilon^*\,(-)^{j+m}\eta'\,\epsilon\,(-)^{j'+m'}\;\theta(m)\theta(m') \\
&\qquad \times \sum_k \left\{ \eta\epsilon^*(-)^{j-m}\, V_{j\,-m\,k} + V_{j\,m\,k} \right\} \\
&\qquad\qquad\qquad \times \left\{ \eta'\epsilon(-)^{j'-m'}\, V^*_{j'\,-m'\,k} + V^*_{j'\,m'\,k} \right\}
\end{aligned} \tag{26}$$

In conclusion, we must have, for non-negative values of m and m',

$$ {}^{\epsilon}\rho^{\,j\,j'}_{-m\,-m'} = \eta\eta'(-)^{j-j'}(-)^{m-m'}\; {}^{\epsilon}\rho^{\,j\,j'}_{m\,m'} \tag{27}$$

a remarkable result.

Similarly, we can show that the same formula holds for the original density matrix. The spin density matrix [see (23)] can be written

$$\begin{aligned}
\rho^{jj'}_{mm'} &= \sum_k V_{jmk}\; V^*_{j'm'k} \\
&= \sum_{\substack{\lambda_1\lambda_2;\lambda_a\,\lambda_b \\ j_c\lambda_c\,j_d\lambda_d;j'_c\lambda'_c\,j'_d\lambda'_d}} \langle q_3, j\,m|\mathcal{M}_X|q_c, j_c\lambda_c; q_d, j_d\lambda_d\rangle \langle q_3, j'\,m'|\mathcal{M}_X|q_c, j'_c\lambda'_c; q_d, j'_d\lambda'_d\rangle^* \\
&\qquad \times \langle q_1, j_1\lambda_1; q_c, j_c\lambda_c|\mathcal{M}_a|q_a, j_a\lambda_a\rangle \langle q_1, j_1\lambda_1; q_c, j'_c\lambda'_c|\mathcal{M}_a|q_a, j_a\lambda_a\rangle^* \\
&\qquad \times \langle q_2, j_2\lambda_2; q_d, j_d\lambda_d|\mathcal{M}_b|q_b, j_b\lambda_b\rangle \langle q_2, j_2\lambda_2; q_d, j'_d\lambda'_d|\mathcal{M}_b|q_b, j_b\lambda_b\rangle^*
\end{aligned} \tag{28}$$

and apply the product of appropriate reflection operators for $\mathcal{M}_X$, $\mathcal{M}_a$ and $\mathcal{M}_b$ to find

$$\begin{aligned}\rho^{jj'}_{mm'} = \eta\eta'(-)^{j-j'}(-)^{m-m'} \sum_{\substack{\lambda_1\lambda_2;\lambda_a\,\lambda_b\\ j_c\lambda_c\,j_d\lambda_d;j'_c\lambda'_c\,j'_d\lambda'_d}} \langle q_3, j\ -m|\mathcal{M}_X|q_c, j_c - \lambda_c; q_d, j_d - \lambda_d\rangle \\ \times \langle q_3, j'\ -m'|\mathcal{M}_X|q_c, j'_c - \lambda'_c; q_d, j'_d - \lambda'_d\rangle^* \\ \times \langle q_1, j_1 - \lambda_1; q_c, j_c - \lambda_c|\mathcal{M}_a|q_a, j_a - \lambda_a\rangle\langle q_1, j_1 - \lambda_1; q_c, j'_c - \lambda'_c|\mathcal{M}_a|q_a, j_a - \lambda_a\rangle^* \\ \times \langle q_2, j_2 - \lambda_2; q_d, j_d - \lambda_d|\mathcal{M}_b|q_b, j_b - \lambda_b\rangle\langle q_2, j_2 - \lambda_2; q_d, j'_d\lambda'_d|\mathcal{M}_b|q_b, j_b - \lambda_b\rangle^*\end{aligned} \tag{29}$$

The helicities are summed over internally, so that we conclude

$$\rho^{\,j\,j'}_{-m\,-m'} = \eta\eta'(-)^{j-j'}(-)^{m-m'}\ \rho^{\,j\,j'}_{m\,m'} \tag{30}$$

which is the same as (27).

We shall derive another useful formula involving density matrices. From (23), we find

$$\begin{aligned}{}^{\epsilon}\rho^{jj'}_{mm'} &= \theta(m)\theta(m')\sum_k \left\{V_{j\,m\,k} + \eta\,\epsilon^*\,(-)^{j-m}\,V_{j\,-m\,k}\right\} \\ &\qquad\qquad \times \left\{V^*_{j'\,m'\,k} + \eta'\,\epsilon\,(-)^{j'-m'}\,V^*_{j'\,-m'\,k}\right\} \\ &= \theta(m)\theta(m')\Big\{\rho^{\,j\,j'}_{m\,m'} + \eta\,\epsilon^*\,(-)^{j-m}\,\rho^{\,j\,j'}_{-m\,m'} \\ &\qquad + \eta'\,\epsilon\,(-)^{j'-m'}\,\rho^{\,j\,j'}_{m\,-m'} + \eta\eta'(-)^{j-j'}(-)^{m-m'}\,\rho^{\,j\,j'}_{-m\,-m'}\Big\}\end{aligned}$$

$$(30) \rightarrow = 2\theta(m)\theta(m')\left\{\rho^{\,j\,j'}_{m\,m'} + \eta'\,\epsilon\,(-)^{j'-m'}\,\rho^{\,j\,j'}_{m\,-m'}\right\} \tag{31a}$$

$$(30) \rightarrow = 2\theta(m)\theta(m')\left\{\rho^{\,j\,j'}_{m\,m'} + \eta\,\epsilon^*\,(-)^{j-m}\,\rho^{\,j\,j'}_{-m\,m'}\right\} \tag{31b}$$

3 Naturality of the Exchanged Particles

We note that, from (5) and if $\eta = (-)^j$,

$$|\epsilon j\,0\rangle = |j\,0\rangle, \qquad \text{if} \quad \epsilon = +1 \tag{32a}$$

$$= 0, \qquad \text{if} \quad \epsilon = -1 \tag{32b}$$

For a negative reflectivity ϵ, the $m = 0$ states are not allowed, see (32b). The reflectivity quantum number ϵ has been defined so that it coincides with the naturality of the exchanged particles in Reaction (1). One can prove this by noting that the meson production vertex is in reality a time-reversed process $X^0 \to c + d$ in which a state of arbitrary spin-parity j^η decays into a Reggeon (or the Pomeron) and another Reggeon (or the Pomeron)

$$j^\eta \to \text{Reggeon (or Pomeron)} + \text{Reggeon (or Pomeron)} \tag{33}$$

The helicity-coupling amplitude F^j for this decay[6] is

$$A^j_p(m) \propto F^j_\lambda\, D^{j\,*}_{m\,\lambda}(\phi_p, \theta_p, 0) \tag{34}$$

where λ is the difference in the helicity of the exchanged particles, so that $\lambda = \lambda_c - \lambda_d$. The subscript p stands for the 'production' variables. m is the z-component of spin j in the rest frame. Our choice of the production coordinate system is that the particles c and d are aligned along the z-axis, i.e. $\theta_p = \phi_p = 0$. So we have, since the D^J-function is zero unless $m = \lambda$,

$$A_p^j(\lambda) \propto F_\lambda^j \tag{35}$$

This shows that the production amplitude is simply given by the helicity-coupling amplitude itself.

From the parity conservation in the decay, one finds, introducing the naturalities

$$\nu = \eta(-)^j \quad \text{and} \quad \bar{\nu} = \eta(-)^{-j} \tag{36}$$

we obtain

$$F_\lambda^j = \nu_j \bar{\nu}_c \bar{\nu}_d \, F_{-\lambda}^j \tag{37}$$

The helicity-coupling amplitude F_λ^j is zero, if $\nu_j \bar{\nu}_c \bar{\nu}_d = -1$ *and* λ is zero. We now compare the formula (37) above with (32a) and (32b); we see that the reflectivity quantum number ϵ is in fact the product of three naturalities

$$\epsilon = \nu_j \bar{\nu}_c \bar{\nu}_d \tag{38}$$

so that

$$F_\lambda^j = \epsilon \, F_{-\lambda}^j \tag{39}$$

and that $F_0^j = 0$ if $\epsilon = -1$. F_0^j is nonzero in general if $\epsilon = -1$. This is consistent with (32).

4 Angular Distributions for $\pi\pi$ Systems

The distribution function as a function of $\Omega = (\theta, \phi)$ has a standard expression in terms of the density matrix

$$I(\Omega) = \sum_{\substack{jm \\ j'm'}} \rho_{m\,m'}^{j\,j'} \, D_{m\,0}^{j\,*}(\phi, \theta, 0) \; D_{m'\,0}^{j'}(\phi, \theta, 0) \tag{40}$$

The angular distribution (40) may be expanded in terms of the moments $H(LM)$ via

$$I(\Omega) = \sum_{LM} \left(\frac{2L+1}{4\pi} \right) H(LM) D_{M0}^{L\,*}(\phi, \theta, 0) \tag{41}$$

with

$$\begin{aligned} H(LM) &= \int d\Omega \, I(\Omega) D_{M0}^L(\phi, \theta, 0) \\ (41) \rightarrow &= \sum_{L'M'} \left(\frac{2L'+1}{4\pi} \right) H(L'M') \int d\Omega \, D_{M'0}^{L'\,*}(\phi, \theta, 0) \, D_{M0}^L(\phi, \theta, 0) \\ (40) \rightarrow &= \sum_{\substack{jm \\ j'm'}} \rho_{m\,m'}^{j\,j'} \int d\Omega \, D_{m'\,0}^{j'}(\phi, \theta, 0) \, D_{M0}^L(\phi, \theta, 0) \, D_{m\,0}^{j\,*}(\phi, \theta, 0) \end{aligned} \tag{42}$$

where we have used (103). Thus we obtain an important result[c]

$$H(LM) = \sum_{\substack{jm \\ j'm'}} \left(\frac{4\pi}{2j+1}\right) \rho^{j\,j'}_{m\,m'} (j'm'\,LM|jm)(j'0\,L0|j0) \tag{43}$$

The normalization integral is

$$H(00) = \int \mathrm{d}\Omega\ I(\Omega) \tag{44}$$

We use the hermiticity of ρ in (43) and (101) to find

$$\begin{aligned}
H^*(LM) &= \sum_{\substack{jm \\ j'm'}} \left(\frac{4\pi}{2j+1}\right) \left(\rho^{j\,j'}_{m\,m'}\right)^* (j'm'\,LM|jm)(j'0\,L0|j0) \\
\rho = \rho^\dagger \text{ and } (101c) \rightarrow &= (-)^M \sum_{\substack{jm \\ j'm'}} \left(\frac{4\pi}{2j'+1}\right) \rho^{j'\,j}_{m'\,m} (jm\,L\ -M|j'm')(j0\,L0|j'0) \\
\big[\{jm\} \leftrightarrow \{j'm'\}\big] \rightarrow &= (-)^M \sum_{\substack{jm \\ j'm'}} \left(\frac{4\pi}{2j+1}\right) \rho^{j\,j'}_{m\,m'} (j'm'\,L\ -M|jm)(j'0\,L0|j0) \\
&= (-)^M\, H(L\ -M)
\end{aligned} \tag{45}$$

We now work out, starting from (43) again,

$$\begin{aligned}
(-)^M\, H(L\ -M) &= \sum_{\substack{jm \\ j'm'}} \left(\frac{4\pi}{j+1}\right) \rho^{jj'}_{mm'}\, (-)^M (j'm'L\ -M|jm)(j'0L0|j0) \\
&= \sum_{\substack{j\,m \\ j'\,m'}} \left(\frac{4\pi}{j+1}\right) \rho^{jj'}_{-m\,-m'} (-)^M\, (j'\ -m'L\ -M|j\ -m)(j'0L0|j0) \\
(M = m' - m) \rightarrow &= \sum_{\substack{j\,m \\ j'\,m'}} \left(\frac{4\pi}{j+1}\right) \rho^{jj'}_{-m\,-m'}\, (-)^{m-m'} \\
&\qquad\qquad \times (j'\ -m'L\ -M|j\ -m)(j'0L0|j0) \\
(j' + L - j = \text{even}) \rightarrow &= \sum_{\substack{j\,m \\ j'\,m'}} \left(\frac{4\pi}{j+1}\right) \rho^{jj'}_{-m\,-m'}\, (-)^{m-m'}\, (j'\,m'L\,M|j\,m)(j'0L0|j0) \\
&= \sum_{\substack{j\,m \\ j'\,m'}} \left(\frac{4\pi}{j+1}\right) \eta\eta'(-)^{j-j'} \rho^{jj'}_{m\,m'}\, (j'\,m'L\,M|j\,m)(j'0L0|j0) \\
&= H(LM)
\end{aligned} \tag{46}$$

where we have used the fact that $\eta(-)^j = \eta'(-)^{j'} = +1$ for $\pi\pi$ systems. So, together with (45), we conclude that $H(LM)$ must be real. The angular distribution can now be recast

[c] Our notation for the Clebsch-Gordan coefficients is as follows:
$(j_1 m_1\, j_2 m_2|j_2 m_3)$ is equal to $\langle j_1 j_2\, m_1 m_2 | j_1 j_2 j_3 m_3\rangle$ in the PDG Book.

into

$$
\begin{aligned}
I(\Omega) &= \sum_{LM} \left(\frac{2L+1}{4\pi}\right) H(LM) D^{L\,*}_{M0}(\phi,\theta,0) \\
(M>0) \rightarrow &= \sum_{LM} \left(\frac{2L+1}{4\pi}\right) \tau(M) d^L_{M0}(\theta)\; H(LM) \exp i\, M\phi
\end{aligned}
\tag{47}
$$

where

$$
\begin{aligned}
\tau(M) &= 2, \quad M>0, \\
&= 1, \quad M=0, \\
&= 0, \quad M<0
\end{aligned}
\tag{48}
$$

Note that all the terms of (47) are now real. Since the D-functions form a complete orthonormal set in the space $\Omega = (\theta, \phi)$, one merely needs to specify a set of the H's to uniquely define an angular distribution.

5 Angular Distributions for $\pi\pi\pi\pi$ Systems

Angular Distributions for 4π Systems are much more complicated, because the phase space spanned by a four-body is complex; it is in fact 8-dimensional. The decay amplitudes can be classified according to the decay modes: $A_{31} = 3(X^0) \rightarrow$ (3-body) $+\,\pi$, (3-body) $\rightarrow$ (2-body) $+\,\pi$; or $A_{22} = 3(X^0) \rightarrow$ (2-body) $+$ (2-body).

We write down the amplitudes in full, denoting the 3-body amplitude by $|s\,\lambda_s\rangle$ and the following 2-body amplitude by $|\ell\,\lambda\rangle$,

$$
|jm\rangle \rightarrow |s\,\lambda_s\rangle + |\pi\rangle, \quad |s\,\lambda_s\rangle \rightarrow |\ell\,\lambda\rangle + |\pi\rangle, \quad |\ell\,\lambda\rangle \rightarrow |2\pi\rangle
\tag{49}
$$

or

$$
A^{j\,s\,\ell}_{31}(m) = F^j_{\lambda_s}\, D^{j\,*}_{m\lambda_s}(\phi_3,\theta_3,0)\, F^s_\lambda\, D^{s\,*}_{\lambda_s\,\lambda}(\phi_2,\theta_2,0)\, F^\ell_0\, D^{\ell\,*}_{\lambda\,0}(\phi_1,\theta_1,0)
\tag{50}
$$

From parity conservation, we have

$$
\begin{aligned}
&F^j_{\lambda_s} = \nu_j\,\nu_s\,\eta_0\, F^j_{-\lambda_s}, \quad \nu_j = \eta_j(-)^j \quad \nu_s = \eta_s(-)^s \quad \text{and} \quad \eta_0 = -1 \\
&F^s_\lambda = \nu_s\,\nu_\ell\,\eta_0\, F^s_{-\lambda} \quad \text{and} \quad \nu_\ell = (-)^\ell
\end{aligned}
\tag{51}
$$

where $\eta_0 = -1$ is the negative parity of the bachelor pions in the problem. We note that the sequential decays considered here necessrily require rotations of the coordinate sytems, i.e. the secondary decay is described with the z-axis, $\hat{z}_h$, along the break-up momentum in the appropriate rest frame, with the y-axis along $\hat{z}\times\hat{z}_h$, where $\hat{z}$ is the z-axis in the parent rest frame.

The alternative decay mode is

$$
|jm\rangle \rightarrow |\ell_1\,\lambda_1\rangle + |\ell_2\,\lambda_2\rangle, \quad |\ell_1\,\lambda_1\rangle \rightarrow |2\pi\rangle \quad |\ell_2\,\lambda_2\rangle \rightarrow |2\pi\rangle
\tag{52}
$$

so that

$$
A^{j\,\ell_1\,\ell_2}_{\lambda_1\,\lambda_2}(m) = F^{j\,\ell_1\,\ell_2}_{\lambda_1\,\lambda_2}\, D^{j\,*}_{m\lambda}(\phi_4,\theta_4,0) * F^{\ell_1}_0\, D^{\ell_1\,*}_{\lambda_1\,0}(\phi_5,\theta_5,0) * F^{\ell_2}_0\, D^{\ell_2\,*}_{\lambda_2\,0}(\phi_6,\theta_6,0)
\tag{53}
$$

where $\lambda = \lambda_1 - \lambda_2$. Again, from parity conservation, we must have

$$F^{j\,\ell_1\,\ell_2}_{\lambda_1\,\lambda_2} = \nu_j\,\nu_1\,\nu_2\,F^{j\,\ell_1\,\ell_2}_{-\lambda_1\,-\lambda_2}, \qquad \nu_1 = (-)^{\ell_1}, \quad \nu_2 = (-)^{\ell_2} \tag{54}$$

Once again, the coordinate systems follow the same pattern; the z-axis, $\hat{x}_h$, of the daughter frames are fixed along the break-up momenta, with their y-axes given by $\hat{z} \times \hat{z}_h$, where $\hat{z}$ is the z-axis in the parent rest frame.

The reader will have noted that sequential decays described here entail different rest frames, obtained from the original coordinate system by pure time-like Lorentz tranaformations whioch rotations. Thius is perfectly fine, the final particles are spinless pions. But this technique cannot be applied if the final states have nonzero spins, e.g. $p\bar{p} + \pi\pi$, because the final helicity states cannot be correctly described if they are given different rest frames. We point out that the correct procedure is to use the canonical formalism[6], which ensures that all the final-state helicities are evaluated in the same rest frame.

6 ALICE Setup

For the foreseeable future, the Roman pots are not planned for ALICE detector. We work out, therefore, the case in which the recoil particles 1 and 2 are *not* measured, and so we will need to integrate over their degrees of freedom, to arrive at the formulas appropriate for ALICE.

For the purpose, we start with the transition amplitude as defined in (15) with the particle 3 given in the reflectivity basis. Again assuming factorization holds for the reaction (1), we can rewrite (15)

$$\begin{aligned} {}^{\epsilon}V_{j\,m\,k} = \sum_{j_c\lambda_c;\,j_d\lambda_d} & \langle p_3, \epsilon j\, m|\mathcal{M}_3|p_c, j_c\lambda_c; p_d, j_d\lambda_d\rangle_3 \\ & * \langle p_1, j_1\lambda_1; p_c, j_c\lambda_c|\mathcal{M}_a|p_a, j_a\lambda_a\rangle_{a+b} \\ & * \langle p_2, j_2\lambda_2; p_d, j_d\lambda_d|\mathcal{M}_b|p_b, j_b\lambda_b\rangle_{a+b} \end{aligned} \tag{55}$$

where the subscripts $a+b$ and 3 indicate the Lorentz-invariant amplitudes have be evaluated in two different rest frames, i.e. the first one in the 3RF, while the second and the third ones in $a+b$ rest frame, i.e. the laboratory frame of the ALICE detector. We have used the generic momenta p to emphasize that we are going into two different rest frames.

Referring to Fig. 1, we obtain, in the standard 4-momentum notation,

$$p_a + p_b = p_1 + p_3 + p_2 \tag{56a}$$

$$p_c = p_a - p_1 \tag{56b}$$

$$p_d = p_b - p_2 \tag{56c}$$

$$p_3 = p_c + p_d \tag{56d}$$

and

$$\begin{cases} s = (p_a + p_b)^2 = (p_1 + p_2 + p_3)^2 \\ s_{13} = (p_1 + p_3)^2 \quad \text{and} \quad s_{23} = (p_2 + p_3)^2 \\ t_c = p_c^2 = (p_a - p_1)^2 \quad \text{and} \quad t_d = p_d^2 = (p_b - p_2)^2 \\ m_3^2 = (p_c + p_d)^2 \end{cases} \tag{57}$$

Here c and d are space-like, i.e. $t_c = p_c^2 \leq 0$ and $t_d = p_d^2 \leq 0$. We assume that the 'masses' $\sqrt{s}$, m_a, m_b, m_1, m_3 and m_2 are all fixed and different in general for the problem under consideration.

Let $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_3$ and $\boldsymbol{\kappa}_2$ be the two-dimensional vectors in a plane perpendicular to $\boldsymbol{p}_a$ (defined to be the positive z direction) and/or $\boldsymbol{p}_b$ in the overall CM system. And let q be the one-dimensional momentum along $\boldsymbol{p}_a$ in the overall CM system, and so $-q$ is then directed along the negative $\boldsymbol{p}_a$ or, equivalently, along $\boldsymbol{p}_b$, again in the overall CM system. So, by definition, we have

$$\kappa_i = |\boldsymbol{\kappa}_i|, \quad p_i = |\boldsymbol{p}_i| = \sqrt{\kappa_i^2 + q_i^2} \quad \text{and} \quad \varepsilon_i = \sqrt{p_i^2 + m_i^2} \tag{58}$$

where $i = (a, b, 1, 3, 2)$ for the particles in a 2- to 3-body process, see Fig. 1. Note that p_i now stands for *both* the 4-momentum and the magnitude of the 3-momentum, while ε_i stands for the energy of the particle i in the overall CM system. We define

$$p_i = (\varepsilon_i; \boldsymbol{\kappa}_i, q_i), \qquad \kappa_i \ll q_i \quad \text{where} \quad i = (a, b, 1, 2) \tag{59}$$

where the x and y axes span the the plane perpendicular to $\boldsymbol{p}_a$ and/or $\boldsymbol{p}_b$ and the z axis lies along $\boldsymbol{p}_a$. We can further define the 'transverse mass' w_i as follows[d]

$$w_a = m_a, \quad w_b = m_b, \quad w_i = \sqrt{m_i^2 + \kappa_i^2}, \quad i = 1, 2, 3 \tag{60a}$$

$$\varepsilon_a = \sqrt{p^2 + m_a^2} \simeq p + \frac{m_a^2}{2p}, \quad m_a \ll p \tag{60b}$$

$$\varepsilon_b = \sqrt{p^2 + m_b^2} \simeq p + \frac{m_b^2}{2p}, \quad m_b \ll p \tag{60c}$$

$$\varepsilon_i = \sqrt{\kappa_i^2 + q_i^2 + m_i^2} = \sqrt{w_i^2 + q_i^2}, \quad i = 1, 2, 3 \tag{60d}$$

$$(w_i \ll q_i) \to \simeq q_i + \frac{w_i^2}{2q_i}, \quad i = 1,\ 2\ \text{ only} \tag{60e}$$

In particular, we *do not* assume that $w_3 \ll q_3$. We rewrite all of the momenta once again here, setting p to be the magnitude of the momentum $\boldsymbol{p}_a$ or $\boldsymbol{p}_b$ in the overall CM system,

$$\begin{aligned} p_a &= (\varepsilon_a; \vec{0}, p) \quad \text{and} \quad p_b = (\varepsilon_b; \vec{0}, -p) \\ p_1 &= (\varepsilon_1; \boldsymbol{\kappa}_1, q_1) \\ p_2 &= (\varepsilon_2; \boldsymbol{\kappa}_2, -q_2) \\ p_3 &= (\varepsilon_3; \boldsymbol{\kappa}_3, q_3) \end{aligned} \tag{61}$$

[d] For COMPASS on pp interactions at 190 GeV/c, we have $\sqrt{s} \simeq 18.9$ GeV and $p \simeq 9.40$ GeV. The difference $\varepsilon_a - p = \varepsilon_b - p$ is less than 0.05 GeV. Note that $q_1 \simeq q_2 \lesssim 9.40$ GeV.

where we have set $\boldsymbol{p}_a$ and $\boldsymbol{p}_1$ are parallel in the limit $\kappa_1 = 0$, while $\boldsymbol{p}_b$ and $\boldsymbol{p}_2$ are also parallel, again in the limit $\kappa_2 = 0$; note that the z components of the latter are both negative, and

$$\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3 = 0 \quad \text{and} \quad q_2 = q_1 + q_3 \tag{62}$$

Specifically, we require that, for some fixed value q_0,

$$\begin{gathered} q_0 > 0, \quad q_1 > 0 \quad \text{and} \quad q_2 > 0 \\ q_0 \ll q_1, \quad q_0 \ll q_2 \quad \text{and} \quad q_1 \simeq q_2 \\ -q_0 < q_3 < q_0 \end{gathered} \tag{63}$$

Here q_0 is clearly arbitrary.[e]

We see that

$$\begin{aligned} \sqrt{s} &= \varepsilon_1 + \varepsilon_3 + \varepsilon_2 \\ &\simeq q_1 + \varepsilon_3 + q_2 = 2q_1 + \varepsilon_3 + q_3 \simeq 2q_2 + \varepsilon_3 + q_3 \gtrsim 2q_1 \simeq 2q_2 \\ &\text{assuming that} \quad \varepsilon_3 \ll q_1 \quad \text{and} \quad \varepsilon_3 \ll q_2 \end{aligned} \tag{64}$$

while

$$\sqrt{s} = \varepsilon_a + \varepsilon_b \simeq 2p, \quad m_a \ll p \quad \text{and} \quad m_b \ll p \tag{65}$$

There is a hierarchy of momenta

$$|q_3| \leq q_0 \ll q_1 \simeq q_2 \lesssim p \simeq \frac{1}{2}\sqrt{s} \tag{66}$$

We are now ready to calculate, for $s \to \infty$ and $q_1 \to \infty$ and $q_2 \to \infty$ (but $q_3 = q_2 - q_1$ must remain finite.)

$$\begin{cases} t_c = (p_a - p_1)^2 = (\varepsilon_a - \varepsilon_1)^2 - (p - q_1)^2 - \kappa_1^2 \simeq -\kappa_1^2 \\ t_d = (p_b - p_2)^2 = (\varepsilon_b - \varepsilon_2)^2 - (p - q_2)^2 - \kappa_2^2 \simeq -\kappa_2^2 \end{cases} \tag{67}$$

and, with $m_i \approx \kappa_i \ll q_i$, $(i = 1, 2)$,

$$s_{13} = (p_1 + p_3)^2 = m_1^2 + m_3^2 + 2(\varepsilon_1\varepsilon_3 - q_1 q_3 - \kappa_1\kappa_3) \tag{68a}$$

$$\simeq 2q_1(\varepsilon_3 - q_3) + (m_1^2 + m_3^2 - 2\kappa_1\kappa_3) \tag{68b}$$

$$s_{23} = (p_2 + p_3)^2 = m_2^2 + m_3^2 + 2(\varepsilon_2\varepsilon_3 + q_2 q_3 - \kappa_2\kappa_3) \tag{68c}$$

$$\simeq 2q_2(\varepsilon_3 + q_3) + (m_2^2 + m_3^2 - 2\kappa_2\kappa_3) \tag{68d}$$

where we have used the approximations $\varepsilon_1 \simeq q_1$ and $\varepsilon_2 \simeq q_2$ to arrive at (68b) and (68d). Taking the product of the sub-energies in (68b) and (68d), we see that

$$\begin{aligned} s_{13}s_{23} \simeq{}& 4q_1 q_2 w_3^2 + 2q_1(\varepsilon_3 - q_3)(m_2^2 + m_3^2 - 2\kappa_2\kappa_3) \\ &+ 2q_2(\varepsilon_3 + q_3)(m_1^2 + m_3^2 - 2\kappa_1\kappa_3) \\ &+ (m_1^2 + m_3^2 - 2\kappa_1\kappa_3)(m_2^2 + m_3^2 - 2\kappa_2\kappa_3) \end{aligned} \tag{69a}$$

$$\begin{aligned} s_{13}s_{23} \simeq{}& 4q_1 q_2 w_3^2 + 2q_1(\varepsilon_3 - q_3)(m_2^2 + m_3^2 - 2\kappa_2\kappa_3) \\ &+ 2q_2(\varepsilon_3 + q_3)(m_1^2 + m_3^2 - 2\kappa_1\kappa_3) \end{aligned} \tag{69b}$$

[e] For COMPASS on pp interactions at 190 GeV/c, we may set $q_0 = 1.5$ GeV, approximately 16% of $p \simeq 9.40$ GeV.

Observe that the first term is dominant; it is a product of q_1 and q_2 and is independent of q_3. In order to derive (69b), we have assumed that, from (68b) and (68d),

$$\begin{aligned} m_1^2 + m_3^2 - 2\kappa_1\kappa_3 &\ll 2q_1(\varepsilon_3 - q_3) \\ m_2^2 + m_3^2 - 2\kappa_2\kappa_3 &\ll 2q_2(\varepsilon_3 + q_3) \end{aligned} \tag{70}$$

Assuming that these conditions are satisfied, we are able to drop, from (69a), the term devoid of q_1 or q_2 to obtain (69b). We can rewrite the conditions (70) with the conditions spelled out in (64)

$$\begin{aligned} \varepsilon_3 \ll q_2 \quad &\to \quad 2q_1(\varepsilon_3 - q_3) \ll 2q_1(q_2 - q_3) \\ \varepsilon_3 \ll q_1 \quad &\to \quad 2q_2(\varepsilon_3 + q_3) \ll 2q_2(q_1 + q_3) \end{aligned} \tag{71}$$

Combining (70) and (71), we obtain

$$\begin{aligned} m_1^2 + m_3^2 - 2\kappa_1\kappa_3 \ll 2q_1(\varepsilon_3 - q_3) \ll 2q_1(q_2 - q_3) \\ m_2^2 + m_3^2 - 2\kappa_2\kappa_3 \ll 2q_2(\varepsilon_3 + q_3) \ll 2q_2(q_1 + q_3) \end{aligned} \tag{72}$$

which show that ε_3 and $|q_3|$ cannot be too large or small.

Retaining only the first term in (69) and using (66), we obtain finally,

$$\text{Regge Domain:} \qquad \boxed{s_{13}s_{23} \simeq s w_3^2 = s(m_3^2 + \kappa_3^2)} \tag{73}$$

a well-known relationship.[f] For a given $\sqrt{s}$ and a fixed w_3, we see that $\sqrt{s_{13}}$ and $\sqrt{s_{23}}$ are dependent on each other by a parabola. This is a crucial formula for central production; *it can be used to define the central production*; no other cuts are needed, not even the t cuts. Kaidalov[11] defines the *Regge domain* through the relationship (73). We recapitulate the conditions under which the formulas (67), (69) and (73) are valid:

(1) $m_1 \approx \kappa_1 \approx w_1 \ll p \approx q_1 \simeq \frac{1}{2}\sqrt{s}$
and $-t_c \simeq \kappa_1^2 \approx m_1^2$ remains finite

(2) $m_2 \approx \kappa_2 \approx w_2 \ll p \approx q_2 \approx q_1 \simeq \frac{1}{2}\sqrt{s}$
and $-t_d \simeq \kappa_2^2 \approx m_2^2$ remains finite

(3) $m_3 \approx \kappa_3 \lesssim w_3$ and $\varepsilon_3^2 = w_3^2 + q_3^2$ where $-q_0 < q_3 < q_0$;
(a) $q_2 = q_1 + q_3$; (b) $q_0 \ll q_1 \simeq q_2 \lesssim p \simeq \frac{1}{2}\sqrt{s}$; (c) $\varepsilon_3 \ll q_1 \simeq q_2$;
(d) $m_1^2 + m_3^2 - 2\kappa_1\kappa_3 \ll 2q_1(\varepsilon_3 - q_3) \ll 2q_1(q_2 - q_3)$;
(e) $m_2^2 + m_3^2 - 2\kappa_2\kappa_3 \ll 2q_2(\varepsilon_3 + q_3) \ll 2q_2(q_1 + q_3)$

[f] For COMPASS on pp interactions at 190 GeV/c, we have $\sqrt{s} \simeq 18.9$ GeV and $w_3 < 2.1$ GeV. So in the case in which the sub-energies are approximately equal, we find $\sqrt{s_{13}} \simeq \sqrt{s_{23}} \lesssim 6.30$ GeV.

It is the third condition, with the subsidiary conditions (b), (c), (d) and (e), which are necessary for the equation (73) to be valid. *We have thus defined the precise conditions for the 'central production of a resonance.'*

We further develop the longitudianl components q_1, q_2, q_3 of the particles 1, 2 and 3 in the overall CM system. From (60e), (62) and (64), we see that

$$\sqrt{s} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \simeq q_1 + \frac{w_1^2}{2q_1} + q_2 + \frac{w_2^2}{2q_2} + \varepsilon_3 \qquad (74a)$$

$$(q_2 = q_1 + q_3) \to \simeq 2q_1 + \frac{w_1^2}{2q_1} + \frac{w_2^2}{2(q_1+q_3)} + \varepsilon_3 + q_3 \qquad (74b)$$

$$\simeq 2q_1 + \frac{w_1^2}{2q_1} + \frac{w_2^2}{2q_1}\left(1 - \frac{q_3}{q_1}\right) + \varepsilon_3 + q_3$$

$$(\text{drop the term with } q_3/q_1) \to \simeq 2q_1 + \frac{w_1^2 + w_2^2}{2q_1} + \varepsilon_3 + q_3 \qquad (74c)$$

So we obtain a quadratic equation in q_1

$$4q_1^2 - 2(\sqrt{s} - \varepsilon_3 - q_3)q_1 + (w_1^2 + w_2^2) \simeq 0 \qquad (75)$$

so that

$$4q_1 \simeq (\sqrt{s} - \varepsilon_3 - q_3) \pm \Big[(\sqrt{s} - \varepsilon_3 - q_3)^2 - 4(w_1^2 + w_2^2)\Big]^{1/2} \qquad (76)$$

The discriminant cannot be negative

$$\sqrt{s} - \varepsilon_3 - q_3 \geq 2(w_1^2 + w_2^2)^{1/2} \qquad (77)$$

The maximum transverse mass for w_3 is thus given by, for $q_3 = 0$,

$$w_3\big|_{\max} = \sqrt{s} - 2(w_1^2 + w_2^2)^{1/2} \qquad (78)$$

Likewise, the effective mass for particle 3 is given by, with $q_3 = 0$ and $\boldsymbol{\kappa}_3 = 0$,

$$n\, m_\pi \leq m_3 \leq m_3\big|_{\max} = w_3\big|_{\max} = \sqrt{s} - 2(w_1^2 + w_2^2)^{1/2} \qquad (79)$$

where $n =$ even (greater than zero), i.e. $2, 4, 6, \cdots$.

Now rewrite (76) as

$$4q_1 \simeq (\sqrt{s} - \varepsilon_3 - q_3)\left\{1 \pm \left[1 - \frac{4(w_1^2 + w_2^2)}{(\sqrt{s} - \varepsilon_3 - q_3)^2}\right]^{1/2}\right\} \qquad (80)$$

If $\sqrt{s} \gg \varepsilon_3 + q_3$ and $\sqrt{s} \gg (w_1^2 + w_2^2)^{1/2}$, typical experimental conditions for COMPASS and ALICE, we have

$$q_1 \simeq \frac{1}{2}(\sqrt{s} - \varepsilon_3 - q_3)\left\{1 - \frac{w_1^2 + w_2^2}{(\sqrt{s} - \varepsilon_3 - q_3)^2}\right\} \qquad (\text{Solution 1}) \qquad (81a)$$

$$\simeq \frac{w_1^2 + w_2^2}{2(\sqrt{s} - \varepsilon_3 - q_3)} \qquad (\text{Solution 2}) \qquad (81b)$$

which shows that q_1 can be given as a function of (ε_3+q_3). Note that there exist *two positive* solutions for q_1 as seen by (81) for a given ε_3 and q_3. But, if $\sqrt{s} \gg \varepsilon_3 + q_3$, then only the first solution (81a) is acceptable, because the second solution for q_1 is likely to be very small, violating the condition that $q_1 \gg \varepsilon_3$ and $q_1 \gg |q_3|$, see (63). Correspondingly, there is *one positive* solution for q_2 as well, through $q_2 = q_1 + q_3$,

$$q_2 \simeq \frac{1}{2}(\sqrt{s} - \varepsilon_3 - q_3)\Big\{1 - \frac{w_1^2 + w_2^2}{(\sqrt{s} - \varepsilon_3 - q_3)^2}\Big\} + q_3 \tag{82a}$$

$$\simeq \frac{1}{2}(\sqrt{s} - \varepsilon_3 + q_3)\Big\{1 - \frac{w_1^2 + w_2^2}{s - 2\sqrt{s}\,\varepsilon_3 - w_3^2}\Big\} \tag{82b}$$

It is instructive to add (81a) and (82a)

$$q_1 + q_2 \simeq \sqrt{s} - \varepsilon_3 - q_3 - \frac{w_1^2 + w_2^2}{(\sqrt{s} - \varepsilon_3 - q_3)} + q_3 \tag{83}$$

or

$$q_1 + q_2 + \varepsilon_3 + \frac{w_1^2 + w_2^2}{(\sqrt{s} - \varepsilon_3 - q_3)} \simeq \sqrt{s} \tag{84}$$

But this should be consistent with (74c), i.e.

$$\frac{w_1^2 + w_2^2}{2q_1} \simeq \frac{w_1^2 + w_2^2}{(\sqrt{s} - \varepsilon_3 - q_3)} \tag{85}$$

which is clearly correct, because $\sqrt{s} \simeq q_1 + q_2 + \varepsilon_3 = 2q_1 + \varepsilon_3 + q_3$. In summary, we have self-consistent solutions for q_1 in (81a) and q_2 in (82b).

We recapitulate the results here for ease of reference. For a given set of ε_3 and q_3, the q_1 and q_2 can be expressed as

$$\boxed{q_1 \simeq \frac{1}{2}(\sqrt{s} - \varepsilon_3 - q_3)\left\{1 - \frac{w_1^2 + w_2^2}{(\sqrt{s} - \varepsilon_3 - q_3)^2}\right\}} \tag{86}$$

and

$$\boxed{q_2 \simeq \frac{1}{2}(\sqrt{s} - \varepsilon_3 + q_3)\left\{1 - \frac{w_1^2 + w_2^2}{s - 2\sqrt{s}\,\varepsilon_3 - w_3^2}\right\}} \tag{87}$$

The allowed range of $q_{1,2}$ is

$$0 \;\leq\; q_{1,2} \;\leq\; q_{1,2}\Big|_{\max} = \;\frac{1}{2}(\sqrt{s} - m_3)\left\{1 - \frac{w_1^2 + w_2^2}{(\sqrt{s} - m_3)^2}\right\} \tag{88}$$

for a given m_3 with $\boldsymbol{\kappa}_3 = 0$ and $q_3 = 0$. The maximum value of m_3 is given by (78) with $\boldsymbol{\kappa}_3 = 0$ and $q_3 = 0$

$$\boxed{m_3\Big|_{\max} = \sqrt{s} - 2(w_1^2 + w_2^2)^{1/2}} \tag{89}$$

The transverse momentum $\boldsymbol{\kappa}_3$ is known, because $\vec{p}_3$ is known. However, the $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ remain unmeasured, although their sum is known through

$$\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3 = 0 \tag{90}$$

The four-momentum transfers $-t_c$ and $-t_d$ are given by $\kappa_1^2 = |\boldsymbol{\kappa}_1|^2$ and $\kappa_2^2 = |\boldsymbol{\kappa}_2|^2$, which remain unmeasured at ALICE, an important information missing for a study of the central production process.

We come finally to the issue which have *NOT* been addressed so far in this note. At ALICE, the 'transverse mass' w_1 and w_2 are unmeasured, since they depend on the transverse momenta $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$

$$w_1^2 = m_1^2 + \kappa_1^2 \quad \text{and} \quad w_2^2 = m_2^2 + \kappa_2^2 \tag{91}$$

In the overall CM system for ALICE, there are two measured planes; the plane 1 is formed by the beam axis (The z-axis) and $\boldsymbol{\kappa}_3$ (or equivalently $\vec{p}_3$) and the plane formed by (90). Note that, by definition, the latter plane is confined to a two-dimensional plane perpendicular to the z-axis, .i.e. in the x-y plane. This is the coordinmate system for the ALICE detector; we shall designate the y-axis to be along the vertical and the x-axis to be along the horizontal plane or, euivalently, the detector plane. Let α_3 ($0 \longrightarrow 2\pi$) be the angle formed by $\boldsymbol{\kappa}_3$ in the x-y plane, measured from the x-axis. Then, the x and y components of $\boldsymbol{\kappa}_3$ are given by

$$\boxed{\kappa_{3x} = \kappa_3 \cos\alpha_3 \quad \text{and} \quad \kappa_{3y} = \kappa_3 \sin\alpha_3} \tag{92}$$

We now impose a *crucial assumption* that $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ which lie in the x-z plane through (90) have the same magnitude, i.e.

$$\boxed{\kappa_1 = \kappa_2 \equiv f\,\kappa_3}, \qquad 0 < f < f_{\max} \tag{93}$$

where $f_{\max}$ could be of the order of 1 but otherwise undetermined. We assume that $\boldsymbol{\kappa}_1 \times \boldsymbol{\kappa}_2$ point along the positive z-axis. So we must have

$$\kappa_1^2 = -t_c = -t_d = \kappa_2^2 \tag{94}$$

Let the angles α_1 and α_2 describe those of $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$, so that $\alpha_2 > \alpha_1$ by definition. We know that, since

$$\alpha \equiv \pi + \alpha_3 - \alpha_1 = \alpha_2 - \pi - \alpha_3$$

and so

$$\kappa_3 = \kappa_1 \cos(\alpha) + \kappa_2 \cos(\alpha) = 2\kappa\cos\alpha = 2f\,\kappa_3\cos\alpha \quad \text{or} \quad 2f\,\cos\alpha = 1 \tag{95}$$

which shows that $\alpha = 60^\circ$ if $f = 1$. We thus find that

$$\boxed{\alpha_1 = \alpha_3 + \pi - \cos^{-1}\left(\frac{1}{2f}\right)} \qquad \text{and} \qquad \boxed{\alpha_2 = \alpha_3 + \pi + \cos^{-1}\left(\frac{1}{2f}\right)} \tag{96}$$

and so

$$\boxed{\kappa_{1x} = \kappa_1 \cos\alpha_1 \quad \text{and} \quad \kappa_{1y} = \kappa_1 \sin\alpha_1} \tag{97}$$

and

$$\boxed{\kappa_{2x} = \kappa_2 \cos\alpha_2 \quad \text{and} \quad \kappa_{2y} = \kappa_2 \sin\alpha_2} \tag{98}$$

We have thus fully specified $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ as a function of $\boldsymbol{\kappa}_3$ and f. The parameter f is unknown, but it is possible that the data could point to an optimum value of f in a scenario in which a range of f's had been tried.

7 Regge Phenomenology

We take formulas from

'High-Energy Particle Diffraction,' by V. Barone and E. Predazzi

Their formulas (5.88) through (5.93) read for a Regge trajectory $\alpha(t)$,

$$\sigma_{\text{tot}} \sim a\, s^{\alpha(0)-1} \qquad \text{where} \qquad \alpha(t) = \alpha(0) + \alpha'\, t \tag{99a}$$

$$\frac{d\sigma_{\text{el}}}{dt} \sim s^{2(\alpha(0)-1)}\, \mathrm{e}^{-b|t|} \qquad \text{where} \qquad b = b_0 + 2\alpha' \ln s \tag{99b}$$

We have assumed here that both the elastic cross sections and the total cross sections through the optical theorem are dominated by a single Reggeon exchange. See a plot of Regge trajectories in Fig. 3.

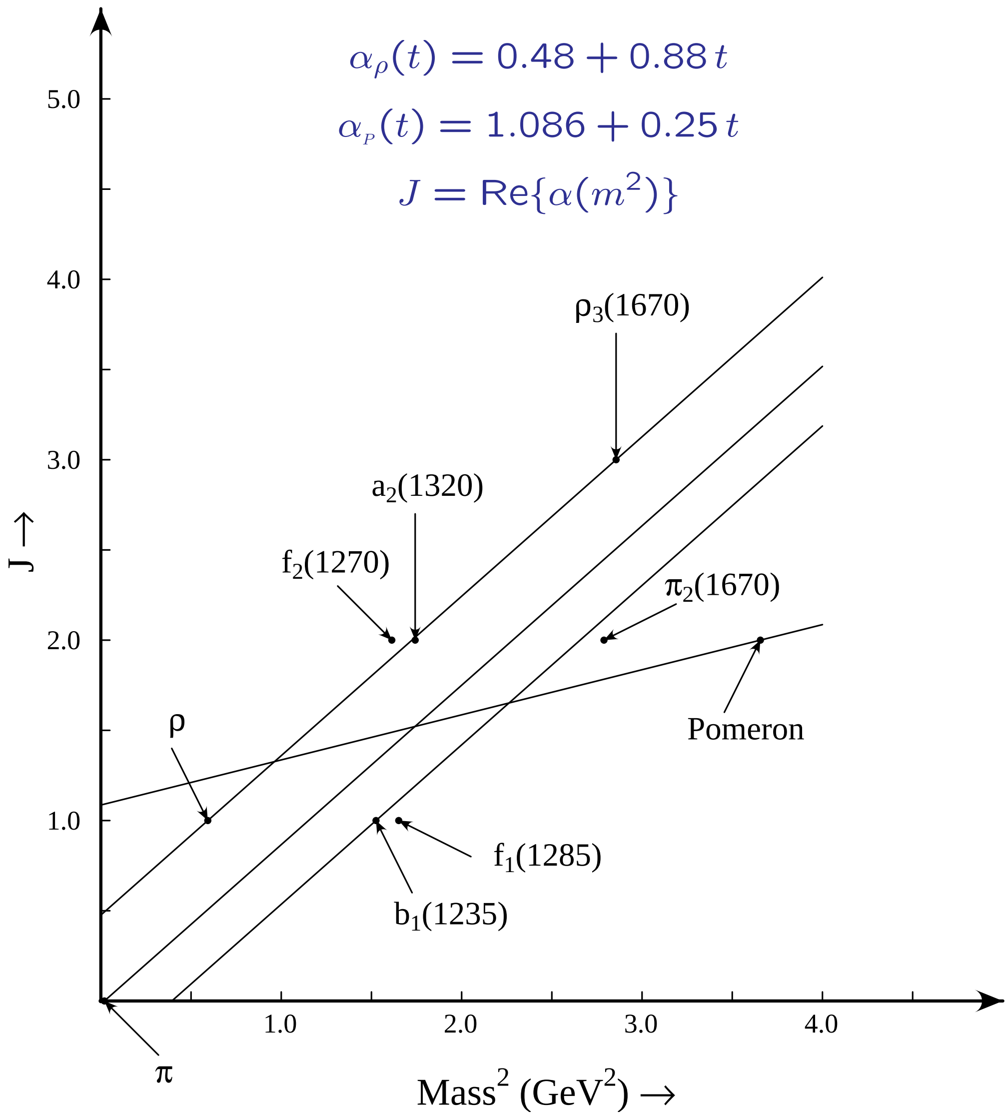

Figure 3: Regge trajectories, including that of the Pomeron

We take

$$\begin{aligned} \alpha_{\boldsymbol{P}}(t) &= 1.086 + 0.25\,t \\ \alpha_\rho(t) &= 0.488 + 0.88\,t \end{aligned} \tag{100}$$

The ratio of the cross sections with the Reggeon set to the $\rho(770)$ divided by that to the Pomeron is as shown in the following table:

$R = \sigma_{\rm tot}[\rho(770)]/\sigma_{\rm tot}(\text{Pomeron})$

Experiment	$p_{\rm beam}(\text{GeV}/c)$	$\sqrt{s}$ (GeV)	R	$\sqrt{s_{13} = s_{23}}$ (GeV) [a]
BNL E852[1]	8	2.4	0.588	2.2
WA102[7]	450	4.5	0.402	3.0
COMPASS[8]	190	5.5	0.356	3.3
ALICE[9]	$3\,500 \times 3\,500$	$7\,000$	0.005	118

[a] We use the formula for the Regge domain, i.e. $\sqrt{s_{13}\, s_{23}} = \sqrt{s}\, w_3$, see (73), with $s_{13} = s_{23}$ and $w_3 = 2.0$ GeV.

This shows that $R = 35.6\%$ at COMPASS but that it reduces to $R = 0.5\%$ at ALICE; so we can safely ignore any Reggeon exchanges other than that of the Pomeron at LHC energies. We have assumed here that the ratio of the double-Pomeron exchanges can be estimated by that of the total cross sections due to the Pomeron exchanges in 2-body-to-2-body reactions. The formula for the Regge domain, shown on the last column above, indicates that the sub-energies, $\sqrt{s_{13}} = \sqrt{s_{23}}$ (GeV) might not be high enough for a reliable central production, even at COMPASS energies. However, the sub-energies for ALICE are at 118 GeV, sufficient for a clean 'double-Pomeron' exchange process leading to the final state with mass at ~ 2.0 GeV.

Appendix A

Clebsch-Gordan Coefficents and D-functions

A few useful Clebsch-Gordan coefficents are:

$$\begin{aligned}(j_1\, m_1\;\, j_2\, m_2|j_3\, m_3) &= (-)^{j_1+j_2-j_3}\,(j_1\;\, -m_1\;\, j_2\;\, -m_2|j_3\;\, -m_3) &&(101a)\\ &= (-)^{j_1+j_2-j_3}\,(j_2\, m_2\;\, j_1\, m_1|j_3\, m_3) &&(101b)\\ &= (-)^{j_1+m_2-j_3}\sqrt{\frac{2j_3+1}{2j_1+1}}\,(j_3\, m_3\;\, j_2\;\, -m_2|j_1\, m_1) &&(101c)\end{aligned}$$

The normalization integrals for the D-functions are,
with $R=(\alpha,\beta,\gamma)$ and $dR = d\alpha\, d\cos\beta\, d\gamma$,

$$\int \mathrm{d}R\, D^{j_1}_{\mu_1 m_1}(R) D^{j_2}_{\mu_2 m_2}(R) D^{j_3\,*}_{\mu_3 m_3}(R) = \frac{8\pi^2}{2j_3+1}(j_1\mu_1 j_2\mu_2|j_3\mu_3)(j_1 m_1 j_2 m_2|j_3 m_3) \tag{102}$$

and, with $\Omega=(\theta,\phi)$ and $\mathrm{d}\Omega = \mathrm{d}\phi\, \mathrm{d}\cos\theta$,

$$\begin{aligned}&\int \mathrm{d}\Omega\, D^{j_1}_{\mu_1 m_1}(\phi,\theta,0) D^{j_2}_{\mu_2 m_2}(\phi,\theta,0) D^{j_3\,*}_{\mu_3 m_3}(\phi,\theta,0)\\ &\qquad = \frac{4\pi}{2j_3+1}(j_1\mu_1 j_2\mu_2|j_3\mu_3)(j_1 m_1 j_2 m_2|j_3 m_3)\end{aligned} \tag{103}$$

provided $m_3 = m_1 + m_2$.

Appendix B

We re-examine the reflectivity formula for the reaction (1) from parity conservation. We do this in the canonical formalism[6], in which the spin components of all the paticles involved are given with respect to the quantization axis, but this is not essential; the helcity formalism is just as well suited for the purpose.

We start with the (12a) written in the conjugate form

$$\langle p, \epsilon j m|\, \Pi_y^\dagger = \epsilon^* (-)^{-2j}\, \langle p, \epsilon j m| \tag{104}$$

and apply to the product of three amplitudes of (15), expressed in terms of the reflectivity eignestates (12a) and (104) in the coordinate system introduced previously

$$\begin{aligned}{}^{\epsilon}V_{j\,m\,k} = \sum_{j_c m_c;\, j_d m_d} &\langle q_3, \epsilon j\, m|\mathcal{M}_X|q_c, \epsilon_c j_c m_c; q_d, \epsilon_d j_d m_d\rangle\\ &\quad * \langle q_1, \epsilon_1 j_1 m_1; q_c, \epsilon_j j_c m_c|\mathcal{M}_a|q_a, \epsilon_a j_a m_a\rangle\\ &\quad * \langle q_2, \epsilon_2 j_2 m_2; q_d, \epsilon_d j_d m_d|\mathcal{M}_b|q_b, \epsilon_b j_b m_b\rangle\end{aligned} \tag{105}$$

and insert the three appropriate products of reflection operators

$$
\begin{aligned}
{}^{\epsilon}V_{j\,m\,k} = &\sum_{\epsilon_c j_c m_c;\,\epsilon_d j_d m_d} \langle q_3, \epsilon j\, m|(\Pi_y)^\dagger\, \Pi_y\, \mathcal{M}_X|q_c, \epsilon_c j_c m_c; q_d, \epsilon_d j_d m_d\rangle \\
&\qquad * \langle q_1, \epsilon_1 j_1 m_1; q_c, \epsilon_j j_c m_c|(\Pi_y^a)^\dagger\, \Pi_y^a\, \mathcal{M}_a|q_a, \epsilon_a j_a m_a\rangle \\
&\qquad * \langle q_2, \epsilon_2 j_2 m_2; q_d, \epsilon_d j_d m_d|(\Pi_y^b)^\dagger\, \Pi_y^b\, \mathcal{M}_b|q_b, \epsilon_b j_b m_b\rangle \\
= &\sum_{\epsilon_c j_c m_c;\,\epsilon_d j_d m_d} \epsilon^*(-)^{-2j}\epsilon_c(-)^{2j_c}\epsilon_d(-)^{2j_d}\langle q_3, \epsilon j\, m|\mathcal{M}_X|q_c, \epsilon_c j_c m_c; q_d, \epsilon_d j_d m_d\rangle \\
&\qquad * \epsilon_1^*(-)^{-2j_1}\;\epsilon_c^*(-)^{-2j_c}\;\epsilon_a(-)^{2j_a} \\
&\qquad\qquad * \langle q_1, \epsilon_1 j_1 m_1; q_c, \epsilon_c j_c m_c|\mathcal{M}_a|q_a, \epsilon_a j_a m_a\rangle \\
&\qquad * \epsilon_2^*(-)^{-2j_2}\;\epsilon_d^*(-)^{-2j_d}\;\epsilon_b(-)^{2j_b} \\
&\qquad\qquad * \langle q_2, \epsilon_2 j_2 m_2; q_d, \epsilon_d j_d m_d|\mathcal{M}_b|q_b, \epsilon_b j_b m_b\rangle \\
= &\,\epsilon^*(-)^{-2j}\epsilon_1^*(-)^{-2j_1}\epsilon_2^*(-)^{-2j_2}\epsilon_a(-)^{2j_a}\epsilon_b(-)^{2j_b} \\
\times &\sum_{\epsilon_c j_c m_c;\,\epsilon_d j_d m_d} \langle q_3, \epsilon j\, m|\mathcal{M}_X|q_c, \epsilon_c j_c m_c; q_d, \epsilon_d j_d m_d\rangle \\
&\qquad * \langle q_1, \epsilon_1 j_1 m_1; q_c, \epsilon_c j_c m_c|\mathcal{M}_a|q_a, \epsilon_a j_a m_a\rangle \\
&\qquad * \langle q_2, \epsilon_2 j_2 m_2; q_d, \epsilon_d j_d m_d|\mathcal{M}_b|q_b, \epsilon_b j_b m_b\rangle \\
= &\,\epsilon^*(-)^{-2j}\epsilon_1^*(-)^{-2j_1}\epsilon_2^*(-)^{-2j_2}\epsilon_a(-)^{2j_a}\epsilon_b(-)^{2j_b}\;{}^{\epsilon}V_{j\,m\,k}
\end{aligned}
\tag{106}
$$

So we obtain

$$
\begin{aligned}
\epsilon^*(-)^{-2j}\epsilon_1^*(-)^{-2j_1}\epsilon_2^*(-)^{-2j_2}\epsilon_a(-)^{2j_a}\epsilon_b(-)^{2j_b} &= +1 \\
\epsilon_a(-)^{2j_a}\epsilon_b(-)^{2j_b} &= \epsilon(-)^{2j}\epsilon_1(-)^{2j_1}\epsilon_2(-)^{2j_2}
\end{aligned}
\tag{107}
$$

so that the product of reflectivities is conserved in the reaction $a + b \to 1 + 3(X^0) + 2$. From this we see that

$$\epsilon(-)^{2j} = \epsilon_1^*\epsilon_2^*\epsilon_a\epsilon_b(-)^{2(j_a+j_b-j_1-j_2)} \tag{108}$$

This shows that the reflectivity ϵ for $3(X^0)$ is related to the reflectivites of particles a, b, 1 and 2.

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