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Relationships involving D-functions and Clebsch-Gordan Coefficients —Version II—

S. U. Chung

Physics Department, Brookhaven National Laboratory, Upton, NY 11973 *

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abstract

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1 D-functions

For the rotation matrix, we use the definition as given in Rose [1], namely,

$$\begin{aligned} D^j_{m'm}(\alpha,\beta,\gamma) &= \langle jm'|\mathrm{e}^{-i\alpha J_z}\mathrm{e}^{-i\beta J_y}\mathrm{e}^{-i\gamma J_z}|jm\rangle \\ &= \mathrm{e}^{im'\alpha}\, d^j_{m'm}(\beta)\,\mathrm{e}^{-im\gamma} \end{aligned} \tag{1}$$

By definition, the matrices $D^j_{m'm}$ are unitary and satisfy the group property:

$$\sum_k D^j_{m'k}(R) D^{j*}_{m\,k}(R) = \sum_k D^j_{k\,m'}(R) D^{j*}_{k\,m}(R) = \delta_{m\,m'} \tag{2}$$

$$D^j_{m'm}(R_2R_1) = \sum_k D^j_{m'k}(R_2) D^j_{km}(R_1)\ . \tag{3}$$

The D-functions are normalized according to

$$\int dR\, D^{j_1*}_{\mu_1 m_1}(R)\, D^{j_2}_{\mu_2 m_2}(R) = \frac{8\pi^2}{2j_1+1}\delta_{j_1j_2}\delta_{\mu_1\mu_2}\delta_{m_1m_2}\ , \tag{4}$$

where $R = R(\alpha,\beta,\gamma)$ and $dR = d\alpha d\cos\beta d\gamma$.

The functions $d^j_{m'm}$ have the following symmetry properties:

$$d^j_{m'm}(\beta) = (-)^{m'-m} d^j_{mm'}(\beta) \tag{5}$$

$$d^j_{m'm}(\beta) = (-)^{m'-m} d^j_{-m'-m}(\beta) \tag{6}$$

$$d^j_{m'm}(\pi-\beta) = (-)^{j+m'} d^j_{m'-m}(\beta) \tag{7}$$

$$d^j_{m'm}(\pi) = (-)^{j-m}\delta_{m',-m} \tag{8}$$

Owing to Eq. (6), the D-functions have the symmetry

$$D^{j*}_{m'm}(\alpha,\beta,\gamma) = (-)^{m'-m} D^j_{-m'-m}(\alpha\beta\gamma)\ . \tag{9}$$

One may use the identity

$$R(\pi+\alpha,\pi-\beta,\pi-\gamma) = R(\alpha,\beta,\gamma)R(0,\pi,0) \tag{10}$$

to show that

$$D^j_{m'm}(\pi+\alpha,\pi-\beta,\pi-\gamma) = (-)^{j-m} D^j_{m'-m}(\alpha,\beta,\gamma) \tag{11}$$

and

$$D^j_{m'm}(\pi+\phi,\pi-\theta,0) = \mathrm{e}^{i\pi j} D^j_{m'-m}(\phi,\theta,0) \tag{12}$$

or taking $\phi \to -\phi$ and using Eq. (9)

$$D^j_{m'm}(\pi-\phi,\pi-\theta,0) = \mathrm{e}^{i\pi j}(-)^{m'+m} D^j_{-m'm}(\phi,\theta,0) \tag{13}$$

the spherical harmonics $Y^\ell_m(\theta,\phi)$ are related to the D-function via

$$D^{\ell *}_{m0}(\phi,\theta,0) = \sqrt{\frac{4\pi}{2\ell+1}} Y^\ell_m(\theta,\phi) \ . \tag{14}$$

The D-functions satisfy the following coupling rule:

$$D^{j_1}_{\mu_1 m_1} D^{j_2}_{\mu_2 m_2} = \sum_{j_3\mu_3 m_3} (j_1\mu_1\, j_2\mu_2|j_3\mu_3)(j_1 m_1\, j_2 m_2|j_3 m_3) D^{j_3}_{\mu_3 m_3} \ . \tag{15}$$

Or, equivalently,

$$D^{j_1}_{\mu_1 m_1} D^{j_3 *}_{\mu_3 m_3} = \sum_{j_2\mu_2 m_2} \left(\frac{2j_2+1}{2j_3+1}\right) (j_1\mu_1 j_2\mu_2|j_3\mu_3)(j_1 m_1 j_2 m_2|j_3 m_3) D^{j_2 *}_{\mu_2 m_2} \ . \tag{16}$$

Modify the formula above

$$\begin{aligned}
D^{j_1}_{\mu_1 m_1} D^{j_3 *}_{\mu_3 m_3} &= \sum_{j_2\mu_2 m_2} \left(\frac{2j_2+1}{2j_3+1}\right) (j_1\mu_1 j_2 -\mu_2|j_3\mu_3)(j_1 m_1 j_2 -m_2|j_3 m_3) D^{j_2 *}_{-\mu_2 -m_2} \\
&= \sum_{j_2\mu_2 m_2} \left(\frac{2j_2+1}{2j_3+1}\right) (j_1\mu_1 j_2 -\mu_2|j_3\mu_3)(j_1 m_1 j_2 -m_2|j_3 m_3) D^{j_2 *}_{-\mu_2 -m_2} \\
&= \sum_{j_2\mu_2 m_2} \left(\frac{2j_2+1}{2j_3+1}\right) \left(\frac{2j_3+1}{2j_1+1}\right)^{1/2} \left(\frac{2j_3+1}{2j_1+1}\right)^{1/2} \\
&\quad (-)^{j_1-m_2-j_3}(j_3\mu_3 j_2\mu_2|j_1\mu_1)(-)^{j_1-\mu_2-j_3}(j_3 m_3 j_2 m_2|j_1 m_1)(-)^{\mu_2-m_2} D^{j_2}_{\mu_2 m_2}
\end{aligned} \tag{17}$$

So we have another important relationship, as an alternative to (16)

$$D^{j_1}_{\mu_1 m_1} D^{j_3 *}_{\mu_3 m_3} = \sum_{j_2 \mu_2 m_2} \left(\frac{2j_2+1}{2j_1+1} \right) (j_2 \mu_3 j_2 \mu_2 | j_1 \mu_1)(j_3 m_3 j_2 m_2 | j_1 m_1) D^{j_2}_{\mu_2 m_2} \tag{18}$$

Recapitulate

$$\boxed{\begin{aligned} D^{j_1}_{\mu_1 m_1} D^{j_3 *}_{\mu_3 m_3} &= \sum_{j_2 \mu_2 m_2} \left(\frac{2j_2+1}{2j_3+1} \right) (j_1 \mu_1 j_2 \mu_2 | j_3 \mu_3)(j_1 m_1 j_2 m_2 | j_3 m_3) D^{j_2 *}_{\mu_2 m_2} \\ D^{j_1}_{\mu_1 m_1} D^{j_3 *}_{\mu_3 m_3} &= \sum_{j_2 \mu_2 m_2} \left(\frac{2j_2+1}{2j_1+1} \right) (j_2 \mu_3 j_2 \mu_2 | j_1 \mu_1)(j_3 m_3 j_2 m_2 | j_1 m_1) D^{j_2}_{\mu_2 m_2} \end{aligned}} \tag{19}$$

It must be emphasized that the two formuals above are in fact identical. One obtains the second by taking the complex conjugate of the first and replacing the subscript 1 by 3. Using the normalization integral of (4), one obtains in addition the integrals involving three D-functions, as follows:

$$\int dR \, D^{j_1}_{\mu_1 m_1}(R) D^{j_2}_{\mu_2 m_2}(R) D^{j_3 *}_{\mu_3 m_3}(R) = \frac{8\pi^2}{2j_3+1} (j_1 \mu_1 j_2 \mu_2 | j_3 \mu_3)(j_1 m_1 j_2 m_2 | j_3 m_3) \; . \tag{20}$$

and

$$\int dR \, D^{j_1}_{\mu_1 m_1}(R) D^{j_2 *}_{\mu_2 m_2}(R) D^{j_3 *}_{\mu_3 m_3}(R) = \frac{8\pi^2}{2j_1+1} (j_3 \mu_3 j_2 \mu_2 | j_1 \mu_1)(j_3 m_3 j_2 m_2 | j_1 m_1) \; . \tag{21}$$

Note that Eq. (21) is a complex conjugate of (20) but with indices 1 and 3 interchanged.

2 Clebsch-Gordan Coefficients

The following relationships involving Clebsch-Gordan coefficients are useful.

$$
\begin{aligned}
(j_1\, m_1\;\, j_2\, m_2 | j_3\, m_3) &= (-)^{j_1+j_2-j_3}\, (j_1\; -m_1\;\, j_2\; -m_2 | j_3\; -m_3) \\
&= (-)^{j_1+j_2-j_3}\, (j_2\, m_2\;\, j_1\, m_1 | j_3\, m_3) \\
&= (-)^{j_1-m_1} \sqrt{\frac{2j_3+1}{2j_2+1}}\, (j_1\, m_1\;\, j_3\; -m_3 | j_2\, m_2) \\
&= (-)^{m_1-j_2+j_3} \sqrt{\frac{2j_3+1}{2j_2+1}}\, (j_1\; -m_1\;\, j_3\, m_3 | j_2\, m_2) \\
&= (-)^{m_1-j_2+j_3} \sqrt{\frac{2j_3+1}{2j_2+1}}\, (j_3\; -m_3\;\, j_1\, m_1 | j_2\; -m_2) \\
&= (-)^{j_1-m_1} \sqrt{\frac{2j_3+1}{2j_2+1}}\, (j_3\, m_3\;\, j_1\; -m_1 | j_2\, m_2) \\
&= (-)^{j_2+m_2} \sqrt{\frac{2j_3+1}{2j_1+1}}\, (j_3\; -m_3\;\, j_2\, m_2 | j_1\; -m_1) \\
&= (-)^{j_2+m_2} \sqrt{\frac{2j_3+1}{2j_1+1}}\, (j_2\; -m_2\;\, j_3\, m_3 | j_1\, m_1) \\
&= (-)^{j_1+m_2-j_3} \sqrt{\frac{2j_3+1}{2j_1+1}}\, (j_3\, m_3\;\, j_2\; -m_2 | j_1\, m_1) \\
&= (-)^{j_1+m_2-j_3} \sqrt{\frac{2j_3+1}{2j_1+1}}\, (j_2\, m_2\;\, j_3\; -m_3 | j_1\; -m_1)
\end{aligned}
\tag{22}
$$

The following sum rules are also useful

$$
\begin{aligned}
&\sum_{m\,m'} (\ell' m' LM|\ell m)\,(\ell' m' L' M'|\ell m) \\
&= \left(\frac{2\ell+1}{2L+1}\right) \sum_{m\,m'} (-)^{\ell'-m'} (\ell m\,\ell' -m'|LM) (-)^{\ell'-m'} (\ell m\,\ell' m'|L' M') \\
&= \left(\frac{2\ell+1}{2L+1}\right) \delta_{LL'}\,\delta_{MM'} \\
&\sum_{L\,M} (2L+1)(\ell' m' LM|\ell m)\,(\ell' n' LM|\ell n) \\
&= \sum_{L\,M} (2L+1)\,\left(\frac{2\ell+1}{2L+1}\right)\,(-)^{\ell'-m'} (\ell m\,\ell' -m'|LM)\,(-)^{\ell'-m'} (\ell n\,\ell' -n'|LM) \\
&= (2\ell+1)\,\delta_{m'\,n'}\,\delta_{m\,n}
\end{aligned} \tag{23}
$$

so that

$$
\begin{aligned}
\sum_{m\,m'} (\ell' m' LM|\ell m)\,(\ell' m' L' M'|\ell m) &= \left(\frac{2\ell+1}{2L+1}\right) \delta_{LL'}\,\delta_{MM'} \\
\sum_{L\,M} (2L+1)\,(\ell' m' LM|\ell m)\,(\ell' n' LM|\ell n) &= (2\ell+1)\,\delta_{m'\,n'}\,\delta_{m\,n}
\end{aligned} \tag{24}
$$

The following relations involving Clebsch-Gordan coefficients can be derived by using the recursion relations for Clebsch-Gordan coefficients [Edmonds [2], p. 39]. In terms of the shorthand notations

$$
\tilde{L} = L(L+1) \quad \text{and} \quad \tilde{J} = J(J+1)
$$

one may write

$$\frac{(J-\frac{1}{2}L1|J\frac{1}{2})}{(J\frac{1}{2}L0|J\frac{1}{2})} = -\frac{2J+1}{\sqrt{\tilde{L}}} \qquad \text{(odd } L \geq 1) \tag{25}$$

$$\frac{(J\frac{3}{2}L0|J\frac{3}{2})}{(J\frac{1}{2}L0|J\frac{1}{2})} = 1-\frac{4\tilde{L}}{4\tilde{J}-3} \qquad = \text{(even } L) \tag{26}$$

$$\frac{(J-\frac{3}{2}L2|J\frac{1}{2})}{(J-\frac{3}{2}L1|J-\frac{1}{2})} = \frac{J+\frac{1}{2}}{\sqrt{\tilde{L}-2}} \qquad \text{(even } L \geq 2) \tag{27}$$

$$\frac{(J1L0|J1)}{(J0L0|J0)} = 1-\frac{\tilde{L}}{2\tilde{J}} \qquad = \text{(even } L) \tag{28}$$

$$\frac{(J-1L2|J1)}{(J0L0|J0)} = -\left[\frac{\tilde{L}}{\tilde{L}-2}\right]^{\frac{1}{2}} \qquad \text{(even } L \geq 2) \tag{29}$$

$$\frac{(J2L0|J2)}{(J0L0|J0)} = 1-\left(\frac{\tilde{L}}{2\tilde{J}}\right)\left(\frac{4\tilde{J}-\tilde{L}-2}{\tilde{J}-2}\right) \qquad \text{(even } L) \tag{30}$$

$$\frac{(J-2L4|J2)}{(J0L0|J0)} = \left[\frac{\tilde{L}(\tilde{L}-6)}{(\tilde{L}-2)(\tilde{L}-12)}\right]^{\frac{1}{2}} \qquad \text{(even } L \geq 4) \tag{31}$$

$$\frac{(J1L1|J2)}{(J0L0|J0)} = -\left[\frac{\tilde{L}-2}{\tilde{L}-6}\right]^{\frac{1}{2}}\left(3-\frac{\tilde{L}}{\tilde{J}}\right) \qquad \text{(even } L \geq 4)\ . \tag{32}$$

References

[1] M.E. Rose, Elementary theory of angular momentum (John Wiley & Sons, Inc., New York, 1957).

[2] A.R. Edmonds, Angular momentum in quantum mechanics (Princeton University Press, Princeton, N.J., 1957).