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One Geometry Problem

S. U. Chung

Physics Department, Brookhaven National Laboratory, Upton, NY 11973 *

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abstract

Consider an arbitrary triangle. Draw lines at 30° from each vertex toward outside of the triangle. The triangle formed by the intersections of each pair of the 30° lines is an equilateral triangle.

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1 Introduction

This short note deals with a proof of the assertion outlined in the abstract.

A derivation is given of the formula for calculating the area of a triangle in Section 1, and the result is used to prove the assertion in Section 2. Section 3 is devoted to working out a possible check of the proof. Finally, certain symmetries exhibited in the problem are pointed out.

2 Area of a Triangle

Consider a triangle with sides a, b and c. The angles opposite each sides are denoted α, β and γ with vertices labeled A_0, B_0 and C_0 (see Fig. 1).

Let A be the area of the triangle, which is given by

$$A = \frac{1}{2}\, ab\, \sin\gamma = \frac{1}{2}\, bc\, \sin\alpha = \frac{1}{2}\, ca\, \sin\beta \tag{1}$$

and the angles are related by

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\, \cos\alpha \\ b^2 &= c^2 + a^2 - 2ca\, \cos\beta \\ c^2 &= a^2 + b^2 - 2ab\, \cos\gamma \end{aligned} \tag{2}$$

Substitute (2) into (1), to obtain

$$\begin{aligned} (4A)^2 &= 4a^2b^2 - (a^2 + b^2 - c^2)^2 \\ &= 4b^2c^2 - (b^2 + c^2 - a^2)^2 \\ &= 4c^2a^2 - (c^2 + a^2 - b^2)^2 \end{aligned} \tag{3}$$

Now add all three above and divide by 3

$$\begin{aligned} (4A)^2 &= 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4) \\ &= a^2(2b^2 - a^2) + b^2(2c^2 - b^2) + c^2(2a^2 - c^2) \end{aligned} \tag{4}$$

This expression for the area is completely symmetric in the variables a, b and c.

The formula (4) can be transformed to a more familiar form. For the purpose, one may write

$$\begin{aligned} -(4A)^2 &= a^4 - 2(b^2+c^2)a^2 + (b^2-c^2)^2 \\ &= a^4 - a^2\left[(b+c)^2 + (b-c)^2\right] + (b+c)^2(b-c)^2 \\ &= \left[a^2-(b+c)^2\right]\left[a^2-(b-c)^2\right] \\ &= (a+b+c)(a-b-c)(a+b-c)(a-b+c) \end{aligned} \tag{5}$$

Introducing a variable

$$s = \frac{1}{2}(a+b+c) \tag{6}$$

one finds an elegant formula

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{7}$$

Once again, note the complete symmetry among the sides in the formula.

3 A Proof

Draw lines at 30° from each vertex toward outside of the triangle. Let the intersection points be denoted A_1, B_1 and C_1, where the point A_1 is on the outside of a and similarly for the others. Now connect the points A_1, B_1 and C_1—the assertion is that this is an equilateral triangle (see Fig.1).

To prove this, one may proceed as follows. By construction, it is clear that the sides of the isosceles triangle with the point A_1 as the vertex and with a as the base is simply given by $a/\sqrt{3}$. Similarly, the same with the point C_1 is $c/\sqrt{3}$. Let f be the length of the line connecting A_1 and C_1. It is given by

$$f^2 = \frac{a^2}{3} + \frac{c^2}{3} - \frac{2\,ac}{3}\cos\delta \tag{8}$$

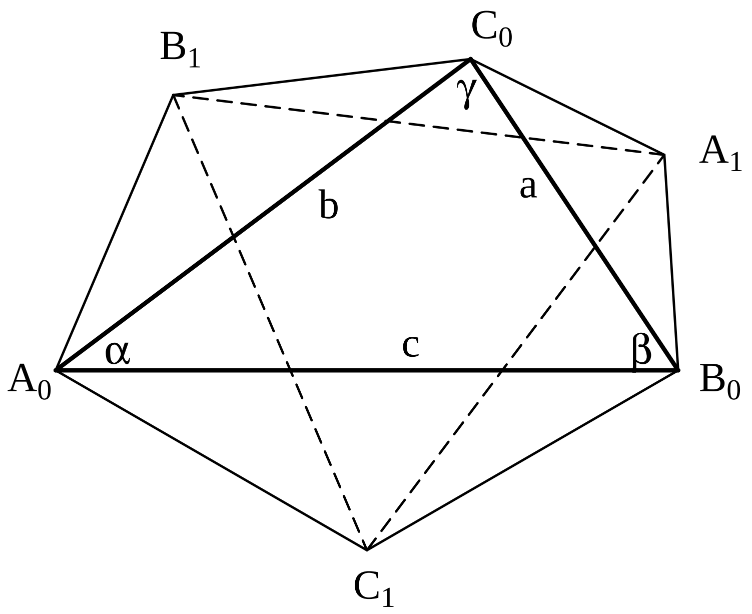

Figure 1: An arbitrary triangle in the problem is denoted with vertices A_0, B_0 and C_0 and sides a, b and c. Lines at 30° from the sides are drawn from each vertex, and the resulting vertices are labeled A_1, B_1 and C_1. The new triangle thus formed is drawn with dashed lines.

where δ is the angle $A_1B_0C_1$ given by

$$\begin{aligned} \cos\delta &= \cos(\beta + \frac{\pi}{3}) \\ &= \cos\beta\cos\frac{\pi}{3} - \sin\beta\sin\frac{\pi}{3} \\ &= \frac{1}{2}\cos\beta - \frac{\sqrt{3}}{2}\sin\beta \end{aligned} \tag{9}$$

or

$$6\,f^2 = 2\,a^2 + 2\,c^2 - 2\,ac\,\cos\beta + 2\sqrt{3}\,ac\,\sin\beta \tag{10}$$

Substituting (2) and (1) into (10), one finds

$$\begin{aligned} 6\,f^2 &= a^2 + b^2 + c^2 + 4\sqrt{3}\,A \\ &= a^2 + b^2 + c^2 + \sqrt{6(a^2b^2 + b^2c^2 + c^2a^2) - 3(a^4 + b^4 + c^4)} \end{aligned} \tag{11}$$

From (4), it is clear that f is completely symmetric in a, b and c. This is the desired result. Consider a special case $a = b = c$, i.e. the original triangle is equilateral. One sees from (11) that $f = a$, so that the triangle formed by A, B and C is again equilateral and is identical to the original.

4 A Check on the Proof

But now one may ask how one can show that the above results are correct. One obvious method is to calculate the area of the hexagon created by the points A, B and C and the vertices of the original triangle. One method is to add the areas of the three isosceles triangles to the area of the original triangle, to obtain

$$4\sqrt{3}\,A_t = a^2 + b^2 + c^2 + \sqrt{6(a^2b^2 + b^2c^2 + c^2a^2) - 3(a^4 + b^4 + c^4)} \tag{12}$$

where A_t is the area of the hexagon. Note that the right-hand sides of (11) and (12) are the same. This indicates that A_t is twice the area of the equilateral triangle with sides f.

The equality of (11) and (12) can be understood by the following argument. The angle $B_0C_1A_0$ is 120° by construction, while the angle $B_1C_1A_1$ is 60°. So the sum of the angles $B_1C_1A_0$ and $B_0C_1A_1$ must be also 60°. Now reflect the triangles $A_0B_1C_1$ and $A_1B_0C_1$ with respect to the cords B_1C_1 and A_1C_1. Since the sides C_1A_0 and B_0C_1 are both equal to $c/sqrt3$, the reflected vertices A_0 and B_0 will meet at a single point E inside of the original triangle. Repeating the same process, it is clear that the vertex C_0 will fall on the same point E (see Fig. 2).

It is clear that, alternatively, the area of the hexagon can be calculated by adding the area of the equilateral triangle formed by f and the three triangles with the sides given by $(a/\sqrt{3}, b/\sqrt{3}, f)$, $(b/\sqrt{3}, c/\sqrt{3}, f)$ and $(c/\sqrt{3}, a/\sqrt{3}, f)$, so that

$$\begin{aligned} 12\,A'_t = {} & \epsilon_c\sqrt{2a^2b^2 + 6(a^2 + b^2)f^2 - (a^4 + b^4 + 9f^4)} \\ & + \epsilon_a\sqrt{2b^2c^2 + 6(b^2 + c^2)f^2 - (b^4 + c^4 + 9f^4)} \\ & + \epsilon_b\sqrt{2c^2a^2 + 6(c^2 + a^2)f^2 - (c^4 + a^4 + 9f^4)} + 3\sqrt{3}\,f^2 \end{aligned} \tag{13}$$

Here ϵ's are normally +1, but if α, β or γ are greater than 120°, then ϵ_a, ϵ_b or ϵ_c must be -1 to take into account of the fact that the cord f lies outside of a vertex of the original triangle.[a] An example is shown in Fig. 4 The assertion is that A_t and A'_t are the same.

Fig. 1 is redrawn with the cords connecting A_0 to A_1, B_0 to B_1 and C_0 to C_1 in Fig. 3. It is remarkable that the three lines intersect at the same point.

[a] This case is treated in the context of a second proof in Section 5 (see Fig. 4).

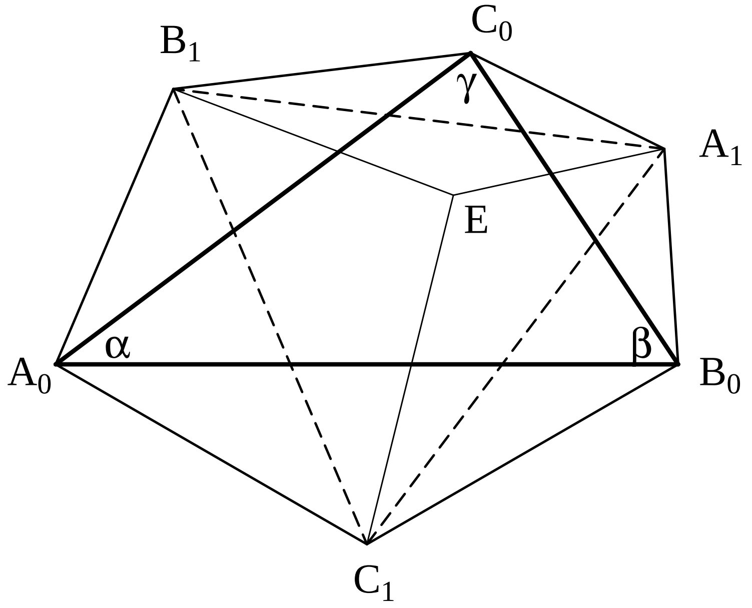

Figure 2: The same figure as Fig. 1, except that the vertices A_0, B_0 and C_0 have been folded into a single point E inside the original triangle.

5 A Second Proof

This proof is purely geometrical and very simple, and it starts out with an equilateral triangle.

Assume an equilateral triangle with vertices A_1, B_1 and C_1 is given as in Fig. 2. Pick an *arbitrary* point E inside the triangle and draw lines to the vertices, as shown. Now, fold out the three triangles A_1EB_1, B_1EC_1 and C_1EA_1 to form new vertices C_0, A_0 and B_0. By construction, the triangle $A_0B_0C_1$ is an isosceles triangle.[b] The same argument holds for the triangles $B_0C_0A_1$ and $C_0A_0B_1$. Again by construction, it is clear that the angle $B_0C_1A_0$ is 120°, twice the angle $B_1C_1A_1$ which is 60°. Therefore, the angles $C_1B_0A_0$ and $C_1A_0B_0$ must be both 30°. Of course, the same argument applies to the vertices A_1 and B_1.

Consider now a point E chosen to be outside of the equilateral triangle. Let the side closest to the point E be A_1B_1 (see Fig. 4). The triangle A_1EB_1 is folded in, whereas the

[b] Note that the same point E is transferred to the points A_0, B_0 and C_0 during the folding-out processes. It is clear, therefore, that $EC_1 = A_0C_1 = B_0C_1$. Note further that, by construction, $\angle EC_1B_1 = \angle A_0C_1B_1$ and $\angle EC_1A_1 = \angle B_0C_1A_1$, so that $\angle A_0C_1B_0 = 2 \times \angle A_1C_1B_1 = 120°$. Since $A_0B_0C_1$ is an isosceles triangle, one sees that $\angle C_1B_0A_0 = \angle C_1A_0B_0 = 30°$.

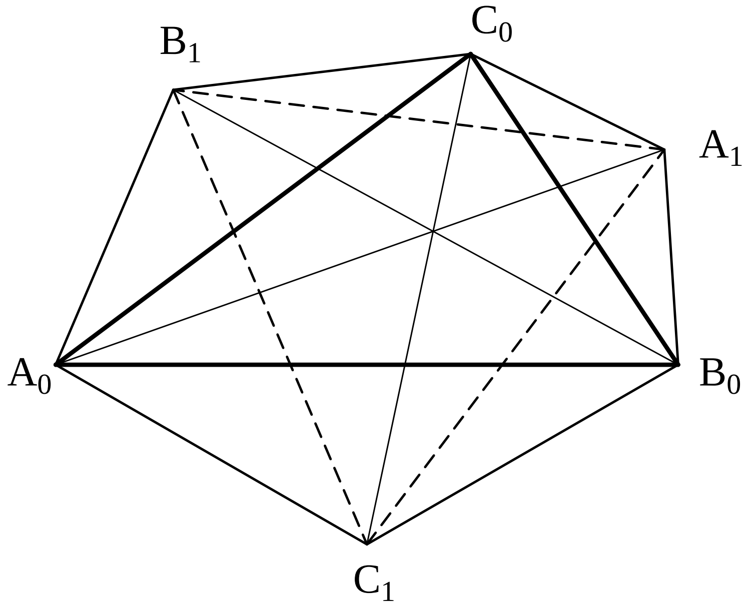

Figure 3: The same figure as Fig. 1, except that the cords are drawn connecting A_0 to A_1, B_0 to B_1 and C_0 to C_1.

triangles B_1EC_1 and C_1EA_1 are folded out, to form new vertices C_0, A_0 and B_0. It might be instructive to scrutinize the folding process in some detail. Let θ be the angle between the cords B_1E and B_1A_1. The folding process is in fact a rotation by 180° around the axis B_1A_1 (a right-hand rotation). The rotated cord B_1C_0 is now at angle 2θ from B_1E. Likewise, let ϕ be the angle between B_1E and B_1C_1. After the rotation, the cord B_1E has been transformed into B_1A_0, which is now at angle 2ϕ from B_1E. The angle $A_0B_1C_0$ is therefore $2(\phi-\theta) = 120°$, since one must have $(\phi - \theta) = 60°$. One sees, incidentally, that this line of argument applies equally to the case in which the point E lies insde the equilateral triangle; one merely assigns a negative number to the angle, if the rotation is left-handed. Note that the angle γ at the vertex C_0 is greater than 120°, as a result of the fact that the point E lies outside the equilateral triangle.

It is instructive to calculate cosine of the angle δ which is defined here as the angle $A_1C_0B_1$. From Fig. 4, one sees that the sum of δ, γ and 60° is equal to 360°. Therefore, one should have

$$\cos\left(\frac{5\pi}{3} - \gamma\right) = \frac{1}{2}\cos\gamma - \frac{\sqrt{3}}{2}\sin\gamma \tag{14}$$

So, following the same argument leading to (11) one finds that the length f given by the

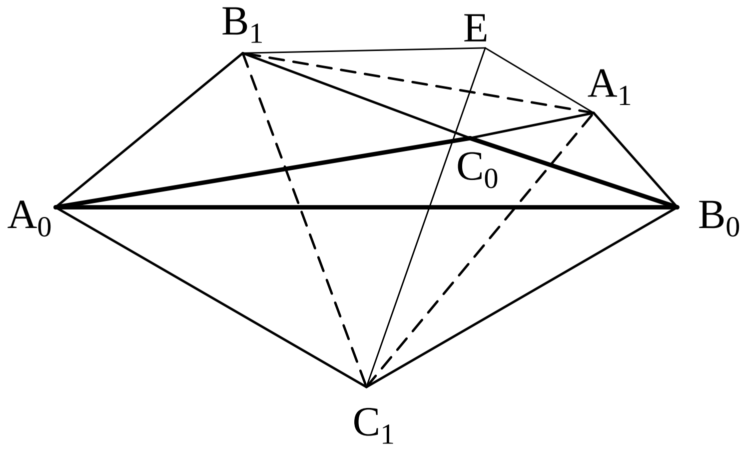

Figure 4: The point E is chosen to be outside the equilateral triangle with vertices A_1, B_1 and C_1. One follows the process outlined in Section 5 to construct the triangle with vertices A_0, B_0 and C_0.

cord A_1B_1 is equal to that of (11).

Finally, it can be shown that the 30° lines need not be drawn outside of the original triangle. An example of such a construction is given in Fig. 5, where the 30° lines are drawn toward the inside of the triangle, from all three vertices A_0, B_0, and C_0. Note that the point E obtained by the same folding process as in Fig. 2 can lie outside the original triangle.

One may ask how the equilateral triangle is related to the triangle with vertices A_0, B_0, and C_0. For the purpose, one may choose a triangle given by $A_1B_0C_1$. The derivation for f (equal to the cord A_1C_1) follows the same line of reasoning which led to (11) One merely needs to calculate the angle δ which is here equal to the angle $A_1B_0C_1$. Rotate the triangle $A_1B_0C_0$ by δ around the point B_0 (right-handed rotation). Let C_0' be the trnasformed vertex C_0 after the rotation. Observe that the angle $A_0B_0C_1'$ is $\delta+\beta$ which is equal to 60°. Therefore, one has

$$\begin{aligned}\cos\delta &= \cos\left(\frac{\pi}{3}-\beta\right)\\ &= \frac{1}{2}\cos\beta + \frac{\sqrt{3}}{2}\sin\beta\end{aligned} \tag{15}$$

so that

$$\begin{aligned}6\,f^2 &= a^2+b^2+c^2-\sqrt{3}\,A\\ &= a^2+b^2+c^2-\sqrt{6(a^2b^2+b^2c^2+c^2a^2)-3(a^4+b^4+c^4)}\end{aligned} \tag{16}$$

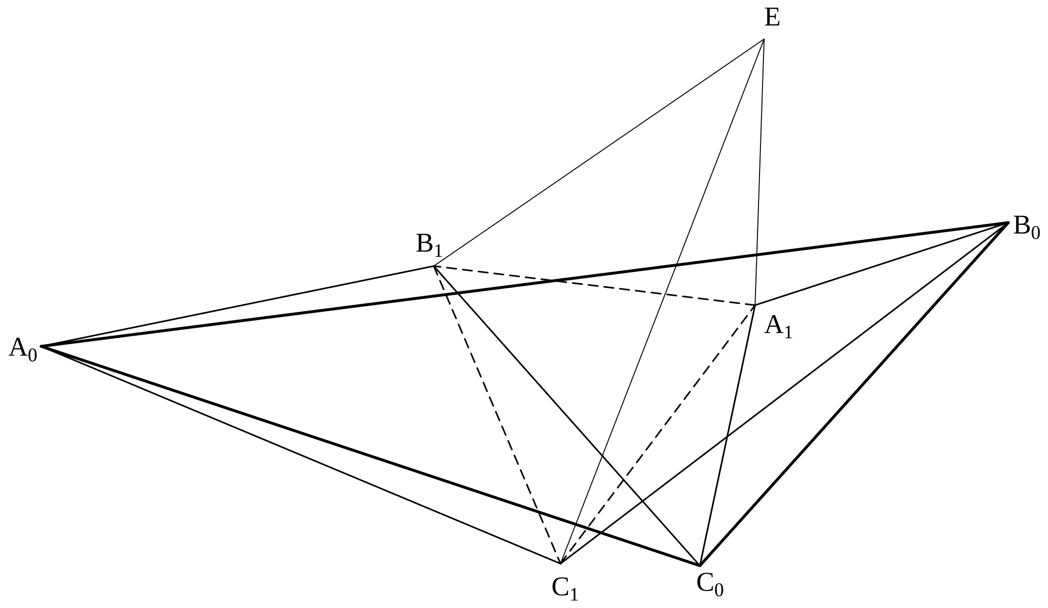

Figure 5: An equilateral triangle with vertices A_1, B_1 and C_1 is formed by drawing the 30° lines toward the inside of the original triangle with vertices A_0, B_0 and C_0.

Note that this expression for f is different from (11) by a sign in front of the square root.

Since the point E has been picked at random, the resulting triangle $A_0B_0C_0$ is likewise a triangle of arbitrary shape—this concludes the second proof.