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Working Draft

Quantum Lorentz Transformations —Version II—

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abstract

Reformulate a work by A. McKerrell so that his formulas conform to our conventions and notations. Reproduce, in addition, much of the work of A. J. Macfarlane. Fill in all the intermediate steps of most of their key results. Add a number of relevant and useful formulas...

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1 Introduction

The purpose of this note is to start with the relativistic spin operator W^μ and construct $\vec{J}$, $\vec{L}$ and $\vec{S}$ operators and explore their actions on single- and two-particle states.

The relevant references are given at the end this note.

2 The Poincaré Group

The general inhomogeneous Lorentz transformation (a, Λ)

$$p'^{\,\mu} = a^\mu + \Lambda^\mu{}_\nu\, p^\nu \quad \text{and} \quad g_{\mu\nu}\, \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\tau = g_{\rho\tau} \tag{2.1}$$

The latter comes from the condition $[p'^{\,2}]_{a=0} = p^2$. Our Lorentz metric $g^{\mu\nu} = g_{\mu\nu}$ has signature $(+,-,-,-)$ so that

$$\begin{aligned} &g^\mu{}_\nu = \delta^\mu{}_\nu \quad \text{and} \quad g_\mu{}^\nu = \delta_\mu{}^\nu \\ &\Lambda_\nu{}^\sigma\, \Lambda^\nu{}_\tau = g^\sigma{}_\tau \quad \text{or} \quad (\Lambda^{-1})^\sigma{}_\nu = \Lambda_\nu{}^\sigma \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} p^\mu &= (E, p^1, p^2, p^3) = (E, p_x, p_y, p_z) \\ p_\mu &= g_{\mu\nu}\, p^\nu = (E, p_1, p_2, p_3) = (E, -p_x, -p_y, -p_z) \end{aligned} \tag{2.3}$$

Let w be the mass associated with p and adopt a notation in which p indicates *both* the four-momentum and the *magnitude* of the 3-momentum, i.e.

$$E^2 = w^2 + p^2, \quad p^2 = p_x^2 + p_y^2 + p_z^2 \tag{2.4}$$

The group multiplication law is

$$\begin{aligned} &(a_1, \Lambda_1)\,(a_2, \Lambda_2) = (a_3, \Lambda_3), \\ &a_3 = a_1 + \Lambda_1\, a_2, \quad \Lambda_3 = \Lambda_1\, \Lambda_2 \end{aligned} \tag{2.5}$$

The corresponding unitary representations follow the same rule

$$\begin{aligned} &U(a, \Lambda) = U(a, 1)\, U(0, \Lambda) \equiv U(a)\, U(\Lambda), \\ &U(a_1, \Lambda_1)\, U(a_2, \Lambda_2) = U(a_3, \Lambda_3) \end{aligned} \tag{2.6}$$

Consider now an homogeneous Lorentz transformation $\Lambda(\vec{\beta})$ $[\vec{\beta} = \vec{p}/E]$ *without* rotation which takes q to q'

$$q'^{\,\mu} = \Lambda^{\mu}{}_{\nu}\, q^{\nu} = \Lambda^{\mu\nu} q_{\nu} \tag{2.7}$$

where

$$\Lambda^{0}{}_{0} = \gamma, \quad \Lambda^{i}{}_{0} = \Lambda^{0}{}_{i} = \eta^{i}, \quad \Lambda^{i}{}_{j} = \delta^{ij} + \frac{\eta^{i}\eta^{j}}{\gamma + 1} \tag{2.8}$$

with $\eta^i = \gamma\beta^i$ and β^i is the velocity of the transformation. Note

$$\gamma^2 = 1 + \eta^2 = \frac{1}{1-\beta^2}, \quad \eta^2 = \eta_x^2 + \eta_y^2 + \eta_z^2 \tag{2.9}$$

Denote $q = (E_q, \vec{q})$ and $q' = (E_q', \vec{q}\,')$ and see that

$$\begin{aligned} E_q' &= \gamma\, E_q + (\vec{\eta}\cdot\vec{q}) \\ \vec{q}\,' &= \vec{q} + \left[E_q + \frac{(\vec{\eta}\cdot\vec{q})}{\gamma+1}\right]\, \vec{\eta} \end{aligned} \tag{2.10}$$

The inverse of the Lorentz transormation $\Lambda(\vec{\beta})$ is given by $\Lambda(-\vec{\beta})$, i.e.

$$(\Lambda^{-1})^{0}{}_{0} = \gamma, \quad (\Lambda^{-1})^{i}{}_{0} = (\Lambda^{-1})^{0}{}_{i} = -\eta^{i}, \quad (\Lambda^{-1})^{i}{}_{j} = \delta^{ij} + \frac{\eta^{i}\eta^{j}}{\gamma+1} \tag{2.11}$$

so that

$$\begin{aligned} E_q &= \gamma\, E_q' - (\vec{\eta}\cdot\vec{q}\,') \\ \vec{q} &= \vec{q}\,' + \left[-E_q' + \frac{(\vec{\eta}\cdot\vec{q}\,')}{\gamma+1}\right]\, \vec{\eta} \end{aligned} \tag{2.12}$$

One can check for consistency by substituting (2.10) into the equations above and see that the equalities hold.

Go over to infinitesimal transformations (a, Λ)

$$a^{\mu} = \epsilon^{\mu} \quad \text{and} \quad \Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \quad \text{or} \quad \Lambda_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu} \tag{2.13}$$

The unitary operators $U(a, \Lambda)$ become

$$\begin{aligned} U(\epsilon) &= U(\epsilon, 1) = 1 - i\,\epsilon_{\mu}\, P^{\mu} \\ U^{-1}(\epsilon) &= U^{-1}(\epsilon, 1) = 1 + i\,\epsilon_{\mu}\, P^{\mu} \\ U(1+\omega) &= U(0, 1+\omega) = 1 - \frac{1}{2}\, i\,\omega_{\mu\nu}\, J^{\mu\nu} \\ U^{-1}(1+\omega) &= U^{-1}(0, 1+\omega) = 1 + \frac{1}{2}\, i\,\omega_{\mu\nu}\, J^{\mu\nu} \end{aligned} \tag{2.14}$$

P and M are Hermitian, and ω and M are antisymmetric That ω is antisymmetric follows from the Lorentz condition (2.1). For infinitesimal Lorentz transformations, it reads

$$\begin{aligned} g_{\rho\tau} &= g_{\mu\nu}\,(g^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})\,(g^{\nu}{}_{\tau} + \omega^{\nu}{}_{\tau}) \\ &= g_{\rho\tau} + \omega_{\rho\tau} + \omega_{\tau\rho} \end{aligned} \tag{2.15}$$

However, it is instructive to explicitly work out ω for (2.8)

$$\omega_{00} = 0, \quad \omega_{0\,i} = \beta^{\,i}, \quad \omega_{i\,0} = -\beta^{\,i}, \quad \omega_{i\,j} = 0 \tag{2.16}$$

to first oder in $\beta^{\,i}$. Note that $\omega_{\mu\nu} = -\omega_{\nu\mu}$ as expected. From (2.5) and (2.6), one sees that

$$U(\Lambda)U(a) = U(\Lambda\,a)\,U(\Lambda) \tag{2.17}$$

so that, with $b = \Lambda\,a$,

$$U^{-1}(\Lambda)\,U(b)\,U(\Lambda) = U(\Lambda^{-1}\,b) \tag{2.18}$$

For b infinitesimally small, one finds

$$\begin{aligned} U(b) &= 1 - i\,b_{\mu}\,P^{\mu} \\ U(\Lambda^{-1}\,b) &= 1 - i\,(\Lambda^{-1}\,b)_{\alpha}\,P^{\alpha} \\ &= 1 - i\,(\Lambda^{-1})_{\alpha}{}^{\mu}\,b_{\mu}\,P^{\alpha} \\ &= 1 - i\,\Lambda^{\mu}{}_{\alpha}\,b_{\mu}\,P^{\alpha} \end{aligned} \tag{2.19}$$

with the identity $(\Lambda^{-1})_{\alpha}{}^{\mu} = \Lambda^{\mu}{}_{\alpha}$ [see (2.2)]. And so one obtains

$$U^{-1}(\Lambda)\,P^{\mu}\,U(\Lambda) = \Lambda^{\mu}{}_{\alpha}\,P^{\alpha} \tag{2.20}$$

Next, observe an obvious identity

$$U^{-1}(\Lambda)\,U(\Sigma)\,U(\Lambda) = U(\Lambda^{-1}\,\Sigma\,\Lambda) \tag{2.21}$$

and set $U(\Sigma) = U(1+\omega)$ [see (2.14)]

$$\begin{aligned} U(\Sigma) &= 1 - \frac{1}{2}\,i\,\omega_{\mu\nu}\,J^{\mu\nu} \\ U(\Lambda^{-1}\,\Sigma\,\Lambda) &= 1 - \frac{1}{2}\,i\,(\Lambda^{-1}\,\omega\,\Lambda)_{\alpha\beta}\,J^{\alpha\beta} \\ &= 1 - \frac{1}{2}\,i\,(\Lambda^{-1})_{\alpha}{}^{\mu}\,\omega_{\mu\nu}\,\Lambda^{\nu}{}_{\beta}\,J^{\alpha\beta} \\ &= 1 - \frac{1}{2}\,i\,\Lambda^{\mu}{}_{\alpha}\,\omega_{\mu\nu}\,\Lambda^{\nu}{}_{\beta}\,J^{\alpha\beta} \end{aligned} \tag{2.22}$$

and find

$$U^{-1}(\Lambda)\, J^{\mu\nu}\, U(\Lambda) = \Lambda^{\mu}{}_{\alpha}\, \Lambda^{\nu}{}_{\beta}\, J^{\alpha\beta} \tag{2.23}$$

The commutation relations can be obtained as follows. Observe

$$U(a)U(b) = U(a+b) \tag{2.24}$$

and go over to infinitesimal translations

$$\begin{aligned} U(a) &= 1 - i\,(a\cdot P) - \frac{1}{2}\,(a\cdot P)^2 \\ U(b) &= 1 - i\,(b\cdot P) - \frac{1}{2}\,(b\cdot P)^2 \\ U(a+b) &= 1 - i\,(a\cdot P + b\cdot P) - \frac{1}{2}\,(a\cdot P + b\cdot P)^2 \end{aligned} \tag{2.25}$$

to see that

$$(a\cdot P)\,(b\cdot P) = (b\cdot P)\,(a\cdot P) \tag{2.26}$$

and find

$$[P^{\mu}, P^{\nu}] = 0 \tag{2.27}$$

Consider next an infinitesimal Lorentz transformation $\Lambda = 1 + \omega$. For the purpose, it is convenient to recast ω, as it is antisymmetric,

$$\begin{aligned} \omega^{\mu}{}_{\alpha} &= \frac{1}{2}\,\omega_{\alpha\beta}\,(g^{\mu\alpha}\,\vec{\delta}^{\beta}{}_{\alpha} - g^{\mu\beta}) \\ \omega^{\nu}{}_{\beta} &= \frac{1}{2}\,\omega_{\beta\alpha}\,(g^{\nu\beta}\,\vec{\delta}^{\alpha}{}_{\beta} - g^{\nu\alpha}) \end{aligned} \tag{2.28}$$

where $\vec{\delta}^{\beta}{}_{\alpha}$ acts on the variables on right and replaces the superscript α with β. Applying the first equation above to (2.20) with $\Lambda = 1+\omega$, one sees that

$$[P^{\mu}, J^{\alpha\beta}] = i\,(g^{\mu\alpha}\,P^{\beta} - g^{\mu\beta}\,P^{\alpha}) \tag{2.29}$$

Now apply both of (2.28) to (2.23) with $\Lambda = 1+\omega$, to find

$$[J^{\mu\nu}, J^{\alpha\beta}] = -i\,(g^{\mu\alpha}\,J^{\nu\beta} - g^{\nu\alpha}\,J^{\mu\beta} + g^{\nu\beta}\,J^{\mu\alpha} - g^{\mu\beta}\,J^{\nu\alpha}) \tag{2.30}$$

using the fact that both ω and M are antisymmetric.

Define 'relativistic spin' via

$$W^{\mu} = \frac{1}{2}\,\varepsilon^{\,\mu\,\alpha\,\beta\,\gamma}\,P_{\alpha}\,J_{\beta\gamma} = \frac{1}{2}\,\varepsilon^{\,\mu\,\alpha\,\beta\,\gamma}\,J_{\alpha\beta}\,P_{\gamma} \tag{2.31}$$

where ε is the 4-dimensional totally antisymmetric tensor with $\varepsilon^{0123} = -1$ (so that $\varepsilon_{0123} = +1$). Note that the second relation above comes from (2.29). Since W^μ is a four-vector, one must have

$$U^{-1}(\Lambda)\, W^\mu\, U(\Lambda) = \Lambda^\mu{}_\nu\, W^\nu \tag{2.32}$$

But this formula can be derived independently, noting that

$$\begin{aligned} U^{-1}(\Lambda)\, W^\mu\, U(\Lambda) &= \frac{1}{2}\,\varepsilon^{\mu\alpha\beta\gamma}\, U^{-1}(\Lambda)\, P_\alpha\, U(\Lambda)\, U^{-1}(\Lambda)\, J_{\beta\gamma}\, U(\Lambda) \\ &= \frac{1}{2}\,\varepsilon^{\mu\alpha\beta\gamma}\, \Lambda_\alpha{}^\rho\, \Lambda_\beta{}^\sigma\, \Lambda_\gamma{}^\tau\, P_\rho\, J_{\sigma\tau} \end{aligned} \tag{2.33}$$

One now makes use of an elegant identity [see Appendix]

$$\varepsilon^{\nu\rho\sigma\tau} = \varepsilon^{\mu\alpha\beta\gamma}\, \Lambda_\mu{}^\nu\, \Lambda_\alpha{}^\rho\, \Lambda_\beta{}^\sigma\, \Lambda_\gamma{}^\tau \tag{2.34}$$

to show that

$$\Lambda^\mu{}_\nu\, \varepsilon^{\nu\rho\sigma\tau} = \varepsilon^{\mu\alpha\beta\gamma}\, \Lambda_\alpha{}^\rho\, \Lambda_\beta{}^\sigma\, \Lambda_\gamma{}^\tau \tag{2.35}$$

Sustituting this to (2.33), one obtains the result (2.32). $P^\mu P_\mu$ and $W^\mu W_\mu$ are invariants (i.e. they commute with all the operators), given by w^2 and $-w^2\, j(j+1)$. Define

$$\begin{aligned} J^i &= \frac{1}{2}\,\varepsilon^{ijk}\, J^{jk}; \qquad J_x = J^1 = J^{23}, \quad J_y = J^2 = J^{31}, \quad J_z = J^3 = J^{12} \\ K^i &= J^{0i}; \qquad K_x = K^1 = J^{01}, \quad K_y = K^2 = J^{02}, \quad K_z = K^3 = J^{03}, \end{aligned} \tag{2.36}$$

so that, using $P^i = -P_i$, $J_{ij} = +J^{ij}$, $J_{0i} = -J^{0i}$ and $J_{i0} = +J^{0i}$,

$$\begin{aligned} W^0 &= \vec{P}\cdot\vec{J} = \vec{J}\cdot\vec{P} \\ \vec{W} &= P^0\,\vec{J} - \vec{P}\times\vec{K} = \vec{J}\,P^0 + \vec{K}\times\vec{P} \end{aligned} \tag{2.37}$$

Define 'spin' $\vec{S}(p)$ using the boost operator $U[L(p)]$

$$\begin{aligned} wS^i(p) &= U[L(p)]\, W^i\, U^{-1}[L(p)] \\ &= U^{-1}[L^{-1}(p)]\, W^i\, U[L^{-1}(p)] \\ &= L^{-1}(p)^i{}_\mu\, W^\mu \end{aligned} \tag{2.38}$$

The reason for this particular definition will be made clear in the next section. One can readily verify that, from (2.12),

$$\begin{aligned} wS^i(p) &= W^i - \left[\frac{W^0 + \gamma\, W^0 - (\vec{\eta}\cdot\vec{W})}{\gamma+1}\right]\vec{\eta} \\ &= W^i - \left[\frac{W^0 + (p_\mu\, W^\mu/w)}{\gamma+1}\right]\vec{\eta} \end{aligned} \tag{2.39}$$

Now construct a general spin operator by replacing E and $\vec{p}$ by the translation operators P^0 and $\vec{P}$. Note that $P_\mu W^\mu = 0$ by the definition of W^μ. One may therefore write

$$wS^i = W^i - \frac{1}{P^0 + w} P^i W^0 \tag{2.40}$$

so that

$$w\vec{S} = P^0 \vec{J} - \vec{P} \times \vec{K} - \frac{1}{P^0 + w} \vec{P} \, (\vec{P} \cdot \vec{J}) \tag{2.41}$$

Also

$$\begin{aligned} [P^0, J^i] &= [P^0, S^i] = 0 \\ [J^i, A^j] &= i \, \varepsilon^{ijk} \, A^k \end{aligned} \tag{2.42}$$

where A^i is P^i, J^i, K^i or S^i. And

$$\begin{aligned} &[P^0, K^i] = i \, P^i, \quad [P^i, K^j] = i \, \delta_{ij} \, P^0, \quad [P^i, S^j] = 0 \\ &[S^i, S^j] = i \, \varepsilon^{ijk} \, S^k, \quad [K^i, K^j] = -i \, \varepsilon^{ijk} \, J^k \end{aligned} \tag{2.43}$$

It is instructive to check the commutation rule (2.42) in the case of pure rotation. Start with a point $P_0(1,0,0)$ and perform rotations $R_k(\epsilon_k) = \exp[-i \, \epsilon \, J_k]$ $(k = x, y, z)$ as follows:

$$\begin{aligned} P_1(1, 0, -\epsilon_y) &= R_y(\epsilon_y) \, R_x(\epsilon_x) \, P_0(1,0,0) \\ P_2(1, \epsilon_x \epsilon_y, -\epsilon_y) &= R_x(\epsilon_x) \, R_y(\epsilon_y) \, P_0(1,0,0) \\ P_3(1, \epsilon_x \epsilon_y, 0) &= R_z(\epsilon_x \epsilon_y) \, P_0(1,0,0) \end{aligned} \tag{2.44}$$

where P_1, P_2 and P_3 indicate the positions after rotation. Observe

$$\begin{aligned} P(0, \epsilon_x \epsilon_y, 0) &= P_2(1, \epsilon_x \epsilon_y, -\epsilon_y) - P_1(1, 0, -\epsilon_y) \\ &= P_3(1, \epsilon_x \epsilon_y, 0) - P_0(1,0,0) \end{aligned} \tag{2.45}$$

so that

$$R_x(\epsilon_x) \, R_y(\epsilon_y) - R_y(\epsilon_y) \, R_x(\epsilon_x) = R_z(\epsilon_x \epsilon_y) - 1 \tag{2.46}$$

By expanding $R_k(\epsilon_k) = \exp[-i \, \epsilon \, J_k]$ to second order in ϵ_k, one finds

$$[J_x, J_y] = i \, J_z \tag{2.47}$$

and similarly for others as well.

Define L via

$$\vec{J} = \vec{L} + \vec{S} \tag{2.48}$$

so that

$$w\vec{L} = -(P^0 - w)\vec{J} + \vec{P}\times\vec{K} + \frac{1}{P^0+w}\vec{P}\,(\vec{P}\cdot\vec{J}) \tag{2.49}$$

and

$$\begin{aligned}
&[J^i, L^j] = [L^i, L^j] = i\,\varepsilon^{ijk}\,L^k \\
&[P^0, L^i] = [L^i, S^j] = 0 \\
&[P^i, L^j] = i\,\varepsilon^{ijk}\,P^k, \quad \vec{P}\cdot\vec{L} = 0
\end{aligned} \tag{2.50}$$

Because of the commutation relations, one may rewrite

$$\begin{cases}
W^0 = \vec{J}\cdot\vec{P} \\
\vec{W} = \vec{J}\,P^0 + \vec{K}\times\vec{P} \\
w\vec{S} = \vec{J}\,P^0 + \vec{K}\times\vec{P} - \dfrac{1}{P^0+w}\vec{P}\,(\vec{J}\cdot\vec{P}) \\
w\vec{L} = -\vec{J}\,(P^0 - w) - \vec{K}\times\vec{P} + \dfrac{1}{P^0+w}\vec{P}\,(\vec{J}\cdot\vec{P})
\end{cases} \tag{2.51}$$

When the operators above are applied to the states in an eigenstate of p^μ, then all the operators $(P^0, \vec{P})$ can be replaced by $(E, \vec{p})$ in the formulas above. And then the modified operators are denoted by $W^\mu(p)$, $S^\mu(p)$ and $L^\mu(p)$.

3 Single-Particle States

Define, as in (2.3),

$$\begin{aligned}
&J^1 = J_x, \quad J^2 = J_y, \quad J^3 = J_z, \\
&J_\pm = J_x \pm i\,J_y, \quad \vec{J}^2 = J_x^2 + J_y^2 + J_z^2
\end{aligned} \tag{3.1}$$

The standard representation of angular momentum states are given by

$$\begin{cases}
\vec{J}^2\,|jm\rangle = j(j+1)\,|jm\rangle \\
J_z\,|jm\rangle = m\,|jm\rangle \\
J_\pm\,|jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|jm\pm 1\rangle
\end{cases} \tag{3.2}$$

Rest states $|k, jm\rangle$ for a single particle with mass w and spin j are eigenstates of P, J^2 and J_z with corresponding eigenvalues $k = (w, 0, 0, 0)$, $j(j+1)$ and m. They have the ususal transformation property under rotation R

$$U(R)\,|k, jm\rangle = \sum_{m'} |k, jm'\rangle D^j_{m'm}(R) \tag{3.3}$$

The actions of the relativistic spin

$$W_\mu W^\mu = (W^0)^2 - (\vec{W}\cdot\vec{W}) \tag{3.4}$$

on the rest states are, from (2.51),

$$W_\mu W^\mu \, |k, jm\rangle = -w^2 \, j(j+1) \, |k, jm\rangle \tag{3.5}$$

So $W_\mu W^\mu$ has the eigenvalue indicated above for all, massive and relativistic single-particle states. Note that, using (2.51) again,

$$\begin{cases} \vec{J}^{\,2} \, |k, jm\rangle = j(j+1) \, |k, jm\rangle \\ \vec{S}^{\,2} \, |k, jm\rangle = j(j+1) \, |k, jm\rangle \\ \vec{L}^{\,2} \, |k, jm\rangle = 0 \end{cases} \tag{3.6}$$

Consider now a boost $B_z(p)$ along the z-axis which takes k to $p = (E, 0, 0, p)$, i.e.

$$p^\mu = [B_z(p)]^\mu{}_\nu \, k^\nu, \quad B_z(p) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} \tag{3.7}$$

where $\tanh\alpha = \beta = p/E$, $\cosh\alpha = \gamma = E/w$ and $\sinh\alpha = \gamma\beta = p/w$. Define

$$\mathring{p} = (E, \vec{p}) = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta) \tag{3.8}$$

and

$$\mathring{p}^{\,\mu} = [R(\hat{p})]^\mu{}_\nu \, p^\nu \tag{3.9}$$

where

$$R(\hat{p}) = R_z(\phi) \, R_y(\theta) \tag{3.10}$$

and

$$R_y(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}, \qquad \text{and} \qquad R_z(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

The relevant unitary operators are

$$
\begin{aligned}
U[B_z(p)] &= \exp[-i\,\alpha\,K_z] \\
U[R(\hat{p})] &= U[R_z(\phi)R_y(\theta)] = \exp(-i\,\phi\,J_z)\,\exp(-i\,\theta\,J_y)
\end{aligned}
\tag{3.12}
$$

Define

$$
\begin{aligned}
U[\mathcal{L}(\overset{\circ}{p})] &= U[R(\hat{p})]\,U[B_z(p)] \\
U[L(\overset{\circ}{p})] &= U[R(\hat{p})]\,U[B_z(p)]\,U^{-1}[R(\hat{p})]
\end{aligned}
\tag{3.13}
$$

The canonical and helicity states can now be defined via

$$
\begin{aligned}
|\vec{p}, jm\rangle &= U[L(\overset{\circ}{p})]\,|k, jm\rangle = U[R(\hat{p})]\,U[B_z(p)]\,U^{-1}[R(\hat{p})]\,|k, jm\rangle \\
|\vec{p}, \lambda\rangle &= U[\mathcal{L}(\overset{\circ}{p})]\,|k, \lambda\rangle = U[R(\hat{p})]\,U[B_z(p)]\,|k, \lambda\rangle = U[L(\overset{\circ}{p})]\,U[R(\hat{p})]\,|k, \lambda\rangle
\end{aligned}
\tag{3.14}
$$

where $|k, \lambda\rangle = |k, jm\rangle$ with $\lambda = m$. It is seen that the helicity states are distinguished by the use of λ for the z-component of spin and by omitting j in the ket states (when unambiguous). The helicity and canonical states are related through

$$
|\vec{p}, \lambda\rangle = \sum_m D^j_{m\lambda}(\phi, \theta, 0)|\vec{p}, jm\rangle \tag{3.15}
$$

The inverse is

$$
|\vec{p}, jm\rangle = \sum_\lambda D^{j\,*}_{m\lambda}(\phi, \theta, 0)\,|\vec{p}, \lambda\rangle \tag{3.16}
$$

from orthonormality of the D-functions. The ket states are normalized according to

$$
\begin{aligned}
\langle\vec{p}, jm|\vec{p}\,', j'm'\rangle &= 2E\,(2\pi)^3\,\delta^{(3)}(\vec{p} - \vec{p}\,')\,\delta_{j\,j'}\,\delta_{m\,m'} = \tilde{\delta}(\vec{p} - \vec{p}\,')\,\delta_{j\,j'}\,\delta_{m\,m'} \\
\langle\vec{p}, j\lambda|\vec{p}\,', j'\lambda'\rangle &= 2E\,(2\pi)^3\,\delta^{(3)}(\vec{p} - \vec{p}\,')\,\delta_{j\,j'}\,\delta_{\lambda\,\lambda'} = \tilde{\delta}(\vec{p} - \vec{p}\,')\,\delta_{j\,j'}\,\delta_{\lambda\,\lambda'}
\end{aligned}
\tag{3.17}
$$

Together with the invariant volume element

$$
\tilde{\mathrm{d}}p = \frac{\mathrm{d}^3 p}{(2\pi)^3(2E)} \tag{3.18}
$$

the closure relations can be written

$$
\begin{aligned}
\sum_{jm} \int |\vec{p}, jm\rangle\,\tilde{\mathrm{d}}p\,\langle\vec{p}, jm| &= I \\
\sum_{j\lambda} \int |\vec{p}, j\lambda\rangle\,\tilde{\mathrm{d}}p\,\langle\vec{p}, j\lambda| &= I
\end{aligned}
\tag{3.19}
$$

Let $\mathcal{R}$ be a general rotation $R(\alpha\beta\gamma)$

$$
U(\mathcal{R}) = U[R(\alpha\beta\gamma)] = \exp[-i\,\alpha\,J_z]\,\exp[-i\,\beta\,J_y]\,\exp[-i\,\gamma\,J_z] \tag{3.20}
$$

The canonical and helicity states transform under a pure rotation $\mathcal{R}\,(\vec{p} \rightarrow \vec{p}\,')$

$$
\begin{aligned}
U(\mathcal{R})\,|\vec{p}, jm\rangle &= \sum_{m'} |\mathcal{R}\vec{p}, jm'\rangle\, D^j_{m'm}(\mathcal{R}) \\
U(\mathcal{R})\,|\vec{p}, \lambda\rangle &= |\mathcal{R}\,\vec{p}, \lambda\rangle = |\,\vec{p}\,', \lambda\rangle
\end{aligned}
\tag{3.21}
$$

Canonical states transform as if they were rest-frame states and helicity is conserved under a pure rotation. One needs similar transformation laws for $\vec{S}$ and $\vec{L}$. For the purpose, we go back to the original definition (2.38)

$$
\begin{aligned}
wS^i(p)\,|\vec{p}, jm\rangle &= U[L(\mathring{p})]\,W^i\,U^{-1}[L(\mathring{p})]\,|\vec{p}, jm\rangle \\
&= U[L(\mathring{p})]\,W^i\,|k, jm\rangle = w\,U[L(\mathring{p})]\,J^i\,|k, jm\rangle
\end{aligned}
\tag{3.22}
$$

so that

$$
\begin{cases}
S_z(p)\,|\vec{p}, jm\rangle = m\,|\vec{p}, jm\rangle \\
S_\pm(p)\,|\vec{p}, jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|\vec{p}, jm \pm 1\rangle
\end{cases}
\tag{3.23}
$$

Note that $S_z(p)$ and $S_\pm(p)$ act on $|\vec{p}, jm\rangle$, in exactly the same way that J_z and $J_\pm$ act on the rest states $|k, jm\rangle$. Define

$$
U^S(\mathcal{R}) = U^S[R(\alpha\beta\gamma)] = \exp[-i\,\alpha\,S_z]\,\exp[-i\,\beta\,S_y]\,\exp[-i\,\gamma\,S_z] \tag{3.24}
$$

It is clear that

$$
U^S(\mathcal{R})\,|\vec{p}, jm\rangle = \sum_{m'} |\vec{p}, jm'\rangle\, D^j_{m'm}(\mathcal{R}) \tag{3.25}
$$

Define

$$
U^L(\mathcal{R}) = U^L[R(\alpha\beta\gamma)] = \exp[-i\,\alpha\,L_z]\,\exp[-i\,\beta\,L_y]\,\exp[-i\,\gamma\,L_z] \tag{3.26}
$$

Because $\vec{S}$ and $\vec{L}$ commute, it is clear that

$$
U(\mathcal{R}) = U^S(\mathcal{R})\,U^L(\mathcal{R}) = U^L(\mathcal{R})\,U^S(\mathcal{R}) \tag{3.27}
$$

Noting that

$$
U^L(\mathcal{R}) = [U^S(\mathcal{R})]^{-1}\,U(\mathcal{R}) = U^S(\mathcal{R}^{-1})\,U(\mathcal{R})
$$

and using the group property of the D^j functions, one finds

$$
U^L(\mathcal{R})\,|\vec{p}, jm\rangle = |\mathcal{R}\vec{p}, jm\rangle \tag{3.28}
$$

So, for the canonical states, $U^S(\mathcal{R})$ acts on the spin while leaving the momentum invariant, whereas $U^L(\mathcal{R})$ rotates the momentum but leaves the spin unchanged. This is to be contrasted with the actions of $U(\mathcal{R})$, which act on both the spin and the momentum.

4 Two-Particle States

Consider a system of two spinless particles with momenta p_1 and p_2 and masses w_1 and w_2. Let w be the effective mass of the two-particle system and let $p = p_1 + p_2$ be the total 4-momentum and let q be the breakup momentum in the rest frame of the two-particle system. $\overset{\circ}{\Omega} = (\theta, \phi)$ describes to the direction of $\vec{p}_1$ in the rest frame. We work out the normalization of the product of the two ket states

$$\begin{aligned}&\int \tilde{\mathrm{d}}p_1 \tilde{\mathrm{d}}p_2 \, \langle \vec{p}_1 | \vec{p}_1' \rangle \, \langle \vec{p}_2 | \vec{p}_2' \rangle \\ &= \int \tilde{\mathrm{d}}p_1 \tilde{\mathrm{d}}p_2 \, \tilde{\delta}(\vec{p}_1 - \vec{p}_1') \, \tilde{\delta}(\vec{p}_2 - \vec{p}_2') = 1\end{aligned} \tag{4.1}$$

provided $\vec{p}_1' = \vec{p}_1$ and $\vec{p}_2' = \vec{p}_2$. Adopt a normalization for the two-particle system

$$|p, \overset{\circ}{\Omega}\rangle = a \, |\vec{p}_1\rangle \, |\vec{p}_2\rangle \tag{4.2}$$

by requiring

$$\langle p, \overset{\circ}{\Omega} \, | \, p', \overset{\circ}{\Omega}{}' \rangle = (2\pi)^4 \, \delta^{(4)}(p - p') \, \delta^{(2)}(\overset{\circ}{\Omega} - \overset{\circ}{\Omega}{}') \tag{4.3}$$

It is seen that six variables contained in $\vec{p}_1$ and $\vec{p}_2$ have been transformed into the 4-momentum p and $\overset{\circ}{\Omega}$. To find a, perform a change of variables

$$\begin{aligned}\int \tilde{\mathrm{d}}p_1 \tilde{\mathrm{d}}p_2 \, \langle p, \overset{\circ}{\Omega} \, | \, p', \overset{\circ}{\Omega}{}' \rangle &= a^2 \int \tilde{\mathrm{d}}p_1 \tilde{\mathrm{d}}p_2 \, \langle \vec{p}_1 | \vec{p}_1' \rangle \, \langle \vec{p}_2 | \vec{p}_2' \rangle = a^2 \\ &= (2\pi)^4 \int \tilde{\mathrm{d}}p_1 \tilde{\mathrm{d}}p_2 \; \delta^{(4)}(p - p') \, \delta^{(2)}(\overset{\circ}{\Omega} - \overset{\circ}{\Omega}{}') \\ \text{in RF} \to &= \frac{1}{(4\pi)^2} \int \left(\frac{q^2 \mathrm{d}q \, \mathrm{d}^2 \overset{\circ}{\Omega}}{E_1 E_2} \right) \; \delta(w - E_1 - E_2) \, \delta^{(2)}(\overset{\circ}{\Omega} - \overset{\circ}{\Omega}{}') \\ E_i = \sqrt{w_i^2 + q^2} \to &= \frac{1}{(4\pi)^2} \left(\frac{q}{w} \right)\end{aligned} \tag{4.4}$$

where RF stands the two-particle rest frame, and $i = 1$ or 2.

Define, in the rest frame of the two-particle system of *arbitrary spin*,

$$\begin{aligned}\overset{\circ}{q} &= (E_1, \vec{q}\,) = (E_1, q \sin\theta \cos\phi, q \sin\theta \sin\phi, q \cos\theta) \\ \breve{q} &= (E_2, -\vec{q}\,) = (E_2, -q \sin\theta \cos\phi, -q \sin\theta \sin\phi, -q \cos\theta)\end{aligned} \tag{4.5}$$

Using the boost operators $U[B_z(q)]$ and $U[B_{-z}(q)]$ along the positive and negative z-axis and with $R(\hat{q}) = R_z(\phi)R_y(\theta)$,

$$
\begin{aligned}
U[L(\mathring{q})] &= U[R(\hat{q})]\, U[B_z(q)]\, U^{-1}[R(\hat{q})] \\
U[L(\check{q})] &= U[R(\hat{q})]\, U[B_{-z}(q)]\, U^{-1}[R(\hat{q})]
\end{aligned}
\tag{4.6}
$$

one sets

$$
\begin{aligned}
|\mathring{\Omega}\, m_1 m_2\rangle &= \frac{1}{4\pi}\sqrt{\frac{q}{w}}\; U[L(\mathring{q})]|s_1 m_1\rangle\, U[L(\check{q})]|s_2 m_2\rangle \\
&= \frac{1}{4\pi}\sqrt{\frac{q}{w}}\; |\vec{q}, s_1 m_1\rangle\, | - \vec{q}, s_2 m_2\rangle
\end{aligned}
\tag{4.7}
$$

Adopt a normalization

$$
\langle \mathring{\Omega}\, m_1 m_2 \,|\, \mathring{\Omega}'\, m_1' m_2'\rangle = \delta^{(2)}(\mathring{\Omega} - \mathring{\Omega}')\, \delta_{m_1 m_1'}\, \delta_{m_2 m_2'} \tag{4.8}
$$

It is *very* important to remember here that a factor

$$
(2\pi)^4 \int \mathrm{d}^4 p\, \delta^{(4)}(p - p') = (2\pi)^4 \int \mathrm{d}^4 p\; \delta^{(4)}(p - p_1 - p_2)
$$

is suppressed *but* implied [see (4.3)].

Define

$$
\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}, \quad \vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}, \quad \vec{L} = \vec{L}^{(1)} + \vec{L}^{(2)} \tag{4.9}
$$

and, in analogy to (3.12)

$$
\begin{aligned}
U[R(\Omega)] &= \exp(-i\,\phi\, J_z)\, \exp(-i\,\theta\, J_y) \\
U^S[R(\Omega)] &= \exp(-i\,\phi\, S_z)\, \exp(-i\,\theta\, S_y) \\
U^L[R(\Omega)] &= \exp(-i\,\phi\, L_z)\, \exp(-i\,\theta\, L_y)
\end{aligned}
\tag{4.10}
$$

Introduce a new ket state of a given S

$$
|\mathring{\Omega}\, S m_s\rangle = \sum_{m_1 m_2} (s_1 m_1\, s_2 m_2 | S m_s)\, |\mathring{\Omega}\, m_1 m_2\rangle \tag{4.11}
$$

It follows, from(3.25),

$$
U^S[R(\Omega)]\, |\mathring{\Omega}\, S m_s\rangle = \sum_{m_s'} |\mathring{\Omega}\, S m_s'\rangle\, D^S_{m_s' m_s}(\phi, \theta, 0) \tag{4.12}
$$

Now construct a ket state of a given ℓ and S

$$
|\ell m\, S m_s\rangle = \sum_{m_1 m_2} (s_1 m_1\, s_2 m_2 | S m_s) \int \mathrm{d}\mathring{\Omega}\, Y^\ell_m(\mathring{\Omega}) |\mathring{\Omega}\, m_1 m_2\rangle \tag{4.13}
$$

Apply U^L on the states above and find, from (3.28),

$$\begin{aligned}
U^L[R(\Omega)]\,|\ell m\, Sm_s\rangle &= \sum_{m_1m_2} (s_1m_1\, s_2m_2|Sm_s) \int \mathrm{d}\mathring{\Omega}\, Y^\ell_m(\mathring{\Omega})|R\,\mathring{\Omega}\, m_1m_2\rangle \\
(\mathring{\Omega}' = R\,\mathring{\Omega}) \to &= \sum_{m_1m_2} (s_1m_1\, s_2m_2|Sm_s) \int \mathrm{d}\mathring{\Omega}'\, Y^\ell_m(R^{-1}\mathring{\Omega}')|\mathring{\Omega}'\, m_1m_2\rangle \\
(\text{group property of } D\text{'s}) \to &= \sum_{m'} D^{\ell\,*}_{mm'}(R^{-1}) \sum_{m_1m_2} (s_1m_1\, s_2m_2|Sm_s) \\
&\qquad\qquad \times \int \mathrm{d}\mathring{\Omega}'\, Y^\ell_{m'}(\mathring{\Omega}')|\mathring{\Omega}'\, m_1m_2\rangle \\
(\text{unitary property of } D\text{'s}) \to &= \sum_{m'} |\ell m'\, Sm_s\rangle\, D^\ell_{m'm}(\phi,\theta,0)
\end{aligned} \tag{4.14}$$

Under a general rotation then, the ket states transform according to

$$\begin{aligned}
U[R(\Omega)]\,|\ell m\, Sm_s\rangle &= U^L[R(\Omega)]\, U^S[R(\Omega)]\,|\ell m\, Sm_s\rangle \\
&= \sum_{m'\,m'_s} |\ell m'\, Sm'_s\rangle\, D^\ell_{m'm}(\phi,\theta,0)\, D^S_{m'_s m_s}(\phi,\theta,0)
\end{aligned} \tag{4.15}$$

Construct next a ket state

$$\begin{aligned}
|JM\ell S\rangle &= \sum_{mm_s} (\ell m\, Sm_s|JM)\,|\ell m\, Sm_s\rangle \\
&= \sum_{\substack{mm_s\\ m_1m_2}} (\ell m\, Sm_s|JM)(s_1m_1\, s_2m_2|Sm_s) \int \mathrm{d}\mathring{\Omega}\, Y^\ell_m(\mathring{\Omega})|\mathring{\Omega}\, m_1m_2\rangle
\end{aligned} \tag{4.16}$$

normalized

$$\langle JM\ell S|J'M'\ell'S'\rangle = \delta_{JJ'}\delta_{MM'}\delta_{\ell\ell'}\delta_{SS'} \tag{4.17}$$

It is apparent that

$$U[R(\Omega)]\,|JM\ell S\rangle = \sum_{M'} |JM'\ell S\rangle\, D^J_{M'M}(\phi,\theta,0) \tag{4.18}$$

so that

$$\begin{cases}
\vec{J}^{\,2}\,|JM\ell S\rangle = J(J+1)\,|JM\ell S\rangle \\
\vec{S}^{\,2}\,|JM\ell S\rangle = S(S+1)\,|JM\ell S\rangle \\
\vec{L}^{\,2}\,|JM\ell S\rangle = \ell(\ell+1)\,|JM\ell S\rangle
\end{cases} \tag{4.19}$$

Consider two-particle states in the helicity basis. Define

$$\begin{aligned}
U[\mathcal{L}(\mathring{q})] &= U[R(\hat{q})]\, U[B_z(q)] \\
U[\mathcal{L}(\breve{q})] &= U[R(\hat{q})]\, U[B_{-z}(q)]
\end{aligned} \tag{4.20}$$

so that

$$
\begin{aligned}
|\mathring{\Omega}\,\lambda_1\lambda_2\rangle &= \frac{1}{4\pi}\sqrt{\frac{q}{w}}\;U[\mathcal{L}(\mathring{q})]|s_1\lambda_1\rangle\,U[\mathcal{L}(\breve{q})]|s_2\,{-\lambda_2}\rangle \\
&= \frac{1}{4\pi}\sqrt{\frac{q}{w}}\;|\vec{q},\lambda_1\rangle\,|-\vec{q},-\lambda_2\rangle
\end{aligned}
\tag{4.21}
$$

Construct

$$
|JM\lambda_1\lambda_2\rangle = \sqrt{\frac{2J+1}{4\pi}}\int \mathrm{d}\mathring{\Omega}\,D^{J*}_{M\lambda}(\phi,\theta,0)\,|\mathring{\Omega}\,\lambda_1\lambda_2\rangle \tag{4.22}
$$

normalized by

$$
\langle JM\lambda_1\lambda_2|J'M'\lambda_1'\lambda_2'\rangle = \delta_{JJ'}\delta_{MM'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} \tag{4.23}
$$

where $\lambda = \lambda_1 - \lambda_2$. It can be shown that

$$
U[R(\Omega)]\,|JM\lambda_1\lambda_2\rangle = \sum_{M'}|JM'\lambda_1\lambda_2\rangle\,D^J_{M'M}(\phi,\theta,0) \tag{4.24}
$$

So $|JM\lambda_1\lambda_2\rangle$ is in the helicity basis what $|JM\ell S\rangle$ is in the ℓS-coupling scheme. Recoupling coefficient:

$$
\langle J'M'\ell'S'|JM\lambda_1\lambda_2\rangle = \sqrt{\frac{2\ell+1}{2J+1}}(\ell\,0\,S\lambda|J\lambda)(s_1\lambda_1\,s_2\,{-\lambda_2}|S\lambda)\,\delta_{JJ'}\delta_{MM'} \tag{4.25}
$$

5 Two-body Decays

In the two-body rest frame, the decay amplitude is

$$
\begin{aligned}
A^J_M(\Omega) &= \sum_{\lambda_1\lambda_2}\{\langle\vec{q},\lambda_1|\,\langle-\vec{q},-\lambda_2|\}\mathcal{M}|JM\rangle \\
&= 4\pi\sqrt{\frac{w}{q}}\sum_{\lambda_1\lambda_2}\langle\Omega\,\lambda_1\lambda_2|JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle \\
&= \sqrt{\frac{2J+1}{4\pi}}\sum_{\lambda_1\lambda_2}D^{J*}_{M\lambda}(\phi,\theta,0)\,F^J_{\lambda_1\lambda_2}
\end{aligned}
\tag{5.1}
$$

where

$$
F^J_{\lambda_1\lambda_2} = 4\pi\sqrt{\frac{w}{q}}\,\langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle \tag{5.2}
$$

Alternatively, the decay amplitude may be written

$$
\begin{aligned}
A^J_M(\mathring{\Omega}; m_1 m_2) &= \{\langle \vec{q}, s_1 m_1 |\, \langle -\vec{q}, s_2 m_2 |\} \mathcal{M} | JM\rangle \\
&= \sum_{\ell S} \{\langle \vec{q}, s_1 m_1 |\, \langle -\vec{q}, s_2 m_2 |\} JM\ell S\rangle \langle JM\ell S | \mathcal{M} | JM\rangle \\
&= 4\pi \sqrt{\frac{w}{q}} \sum_{\ell S} \langle \mathring{\Omega}\, m_1 m_2\, | JM\ell S\rangle \langle JM\ell S | \mathcal{M} | JM\rangle \\
&= \sum_{\ell S} \sum_{m m_s} (\ell m\, S m_s | JM)(s_1 m_1\, s_2 m_2 | S m_s)\, Y^\ell_m(\mathring{\Omega})\, G^J_{\ell S}
\end{aligned}
\tag{5.3}
$$

where

$$
G^J_{\ell S} = 4\pi \sqrt{\frac{w}{q}}\, \langle JM\ell S | \mathcal{M} | JM\rangle \tag{5.4}
$$

Consider a special case in which $\Omega = (0, 0)$:

$$
A^J_M(0, 0) = \sqrt{\frac{2J+1}{4\pi}} \sum_{\lambda_1 \lambda_2} F^J_{\lambda_1 \lambda_2}, \quad M = \lambda_1 - \lambda_2 \tag{5.5}
$$

and

$$
\begin{aligned}
A^J_M(0, 0) &= \sum_{\ell S} \sqrt{\frac{2\ell+1}{4\pi}}\, (\ell\, 0\, SM | JM)\, G^J_{\ell S}\, (s_1 m_1\, s_2 m_2 | SM) \\
&= \sum_{\ell S} \sqrt{\frac{2\ell+1}{4\pi}}\, (\ell\, 0\, S\lambda | J\lambda)\, G^J_{\ell S}\, (s_1 \lambda_1\, s_2 -\!\lambda_2 | S\lambda)
\end{aligned}
\tag{5.6}
$$

with $M = \lambda = \lambda_1 - \lambda_2$, so that

$$
F^J_{\lambda_1 \lambda_2} = \sum_{\ell S} \sqrt{\frac{2\ell+1}{2J+1}}\, (\ell\, 0\, S\lambda | J\lambda)(s_1 \lambda_1\, s_2 -\!\lambda_2 | S\lambda)\, G^J_{\ell S} \tag{5.7}
$$

Alternative derivation of the above:

$$
\begin{aligned}
F^J_{\lambda_1 \lambda_2} &= 4\pi \sqrt{\frac{w}{q}}\, \langle JM\lambda_1 \lambda_2 | \mathcal{M} | JM\rangle \\
&= 4\pi \sqrt{\frac{w}{q}} \sum_{\ell S} \langle JM\lambda_1 \lambda_2 | JM\ell S\rangle \langle JM\ell S | \mathcal{M} | JM\rangle \\
&= \sum_{\ell S} \langle JM\lambda_1 \lambda_2 | JM\ell S\rangle\, G^J_{\ell S} \\
&= \sum_{\ell S} \sqrt{\frac{2\ell+1}{2J+1}}\, (\ell\, 0\, S\lambda | J\lambda)(s_1 \lambda_1\, s_2 -\!\lambda_2 | S\lambda)\, G^J_{\ell S}
\end{aligned}
\tag{5.8}
$$

6 Spin-1 Wave Functions in Momentum Space

We shall now confine the state vectors (or wave functions) defined in the momentum space. Homogeneous Lorentz transformations are given by

$$\Lambda = \exp[-\frac{i}{2}\,\omega_{\mu\nu}\,J^{\mu\nu}\,] \tag{6.1}$$

so that $J^{\mu\nu}$ is now an operator imbedded in the momentum space. For infinitesimal transformations, we have

$$\Lambda^{\rho}{}_{\sigma} = \delta^{\rho}{}_{\sigma} - \frac{i}{2}\,\omega_{\mu\nu}\,(J^{\mu\nu})^{\rho}{}_{\sigma} \tag{6.2}$$

As this must be equal to

$$\Lambda^{\rho}{}_{\sigma} = \delta^{\rho}{}_{\sigma} + \omega^{\rho}{}_{\sigma}$$

we can deduce[11] that

$$(J^{\mu\nu})^{\rho}{}_{\sigma} = i\,(g^{\mu\rho}\,\delta^{\nu}{}_{\sigma} - g^{\nu\rho}\,\delta^{\mu}{}_{\sigma}) \tag{6.3}$$

From (2.36), we find

$$(J_x)^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{pmatrix}, \quad (K_x)^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(J_y)^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad (K_y)^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.4}$$

$$(J_z)^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (K_z)^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & 0 & 0 & +i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}$$

It can be verified that the commutations relations (2.42) and (2.43) are satisfied by the 4×4 matrices given above, i.e.

$$[J^i, J^j] = i\,\varepsilon^{ijk}\,J^k, \quad [J^i, K^j] = i\,\varepsilon^{ijk}\,K^k, \quad [K^i, K^j] = -i\,\varepsilon^{ijk}\,J^k \tag{6.5}$$

Note that J^i's are Hermitian but that K^i's are *not*. Derived matrices of interest are

$$\left(J_x^2\right)^\rho{}_\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \quad \left(K_x^2\right)^\rho{}_\sigma = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(J_y^2\right)^\rho{}_\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \quad \left(K_y^2\right)^\rho{}_\sigma = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.6}$$

$$\left(J_z^2\right)^\rho{}_\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \left(K_z^2\right)^\rho{}_\sigma = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

From which we find

$$\left(J^2\right)^\rho{}_\sigma = \left(J_x^2 + J_y^2 + J_z^2\right)^\rho{}_\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & +2 & 0 & 0 \\ 0 & 0 & +2 & 0 \\ 0 & 0 & 0 & +2 \end{pmatrix}$$

$$\left(K^2\right)^\rho{}_\sigma = \left(K_x^2 + K_y^2 + K_z^2\right)^\rho{}_\sigma = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.7}$$

so that, introducing a 4×4 matrix G through

$$G^\mu{}_\nu = g_{\mu\nu} = g^{\mu\nu} \tag{6.8}$$

one sees that

$$J^2 - K^2 = 3\,I, \quad -2J^2 - K^2 = 3\,G, \quad J^2 + K^2 = -I - 2\,G \tag{6.9}$$

where I is the 4×4 identity matrix.

The wave functions at rest for spin-1 are

$$e^\mu(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad e^\mu(\pm 1) = \mp\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \tag{6.10}$$

It can be shown that

$$\begin{cases} J^2\, e(m) = j(j+1)\, e(m), \quad j = 1, \quad m = -1,\ 0,\ +1 \\ J_z\, e(m) = m\, e(m), \quad m = -1,\ 0,\ +1 \\ J_\pm\, e(0) = \sqrt{2}\, e(\pm 1), \quad J_\pm\, e(\mp 1) = \sqrt{2}\, e(0), \quad J_\pm\, e(\pm 1) = 0 \end{cases} \tag{6.11}$$

where $e(m)$ are 4-component column vectors of (6.10). This shows that $e(m)$'s are indeed the spin-1 wave functions imbedded in the 4-momentum space. The 4×4 matrices K^i when applied to $e(m)$ should give the wave functions *away* from the rest frame as described below.

Define

$$\theta^i = \frac{1}{2}\,\varepsilon^{ijk}\,\omega^{jk} = \frac{1}{2}\,\varepsilon^{ijk}\,\omega_{jk}, \quad \alpha^i = \omega_{0\,i} \tag{6.12}$$

so that (6.1) takes on the form

$$\Lambda = \exp[-i\,\vec\theta\cdot\vec J - i\,\vec\alpha\cdot\vec K\,] \tag{6.13}$$

For pure rotations, we can set

$$R^\rho{}_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & R_{jk} \end{pmatrix}, \quad R_{jk} = \exp[-i\,\vec\theta\cdot\vec J_{jk}] \tag{6.14}$$

For example, we have, for a rotation by θ around the y-axis,

$$[R_y(\theta)]_{jk} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{6.15}$$

Likewise, for a boost along the z-axis, we obtain

$$[B_z(p)]^\rho{}_\sigma = \exp[-i\,\alpha\,(K_z)^\rho{}_\sigma\,] = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} \tag{6.16}$$

identical to (3.7). Here $\cosh\alpha = E/w$ and $\sinh\alpha = p/w$.

After a boost along the z-axis, the wave functions (6.10) become

$$e^\mu(\vec{p}, m) = [B_z(p)]^\mu{}_\nu \, e^\nu(m) \tag{6.17}$$

or, writing out the components explicitly,

$$e^\mu(\vec{p}, 0) = \begin{pmatrix} p/w \\ 0 \\ 0 \\ E/w \end{pmatrix}, \qquad e^\mu(\vec{p}, \pm 1) = \mp \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \tag{6.18}$$

Note that $p_\mu \, e^\mu(\vec{p}, m) = 0$ in any Lorentz frame. It is clear that the canonical and helicity states should be given by

$$\begin{cases} e^\mu(\vec{p}, m) = [L(\vec{p})]^\mu{}_\nu \, e^\nu(m) = \left[R(\hat{p}) \, B_z(p) \, R^{-1}(\hat{p})\right]^\mu{}_\nu \, e^\nu(m) \\ e^\mu(\vec{p}, \lambda) = [\mathcal{L}(\vec{p})]^\mu{}_\nu \, e^\nu(\lambda) = [R(\hat{p}) \, B_z(p)]^\mu{}_\nu \, e^\nu(\lambda) \end{cases} \tag{6.19}$$

Given a momentum $p^\mu = (E, \vec{p})$ and $p^\mu p_\mu = w^2$, we can define the modfied form of the relativistic spin W^μ and their derivatives, S^i and L^i [see (2.51)]

$$\begin{cases} W^0(p) = \vec{J} \cdot \vec{p} \\ \vec{W}(p) = E \, \vec{J} + \vec{K} \times \vec{p} \\ w\vec{S}(p) = E \, \vec{J} + \vec{K} \times \vec{p} - \dfrac{\vec{p}\,(\vec{J} \cdot \vec{p})}{E + w} \\ w\vec{L}(p) = -(E - w) \, \vec{J} - \vec{K} \times \vec{p} + \dfrac{\vec{p}\,(\vec{J} \cdot \vec{p})}{E + w} \end{cases} \tag{6.20}$$

Here $(E, \vec{p})$ are *not* operators but merely c-numbers. These are the operators acting on ket states with 4-momenta $(E, \vec{p})$. Observe

$$\begin{aligned} \vec{W}(p) \cdot \vec{W}(p) &= E^2 \, J^2 + (\vec{K} \times \vec{p}) \cdot (\vec{K} \times \vec{p}) + E \, \vec{J} \cdot (\vec{K} \times \vec{p}) + E \, (\vec{K} \times \vec{p}) \cdot \vec{J} \\ &= E^2 \, J^2 + p^2 \, K^2 - (\vec{p} \cdot \vec{K}\,)^2 + 2E \left[\vec{p} \cdot (\vec{J} \times \vec{K}) - i \, (\vec{p} \cdot \vec{K}\,)\right] \end{aligned} \tag{6.21}$$

where we have used a shorthand notation $p^2 = (\vec{p})^2$. So

$$\begin{aligned} -W^\mu(p) \, W_\mu(p) &= -[W^0(p)]^2 + \vec{W}(p) \cdot \vec{W}(p) \\ &= E^2 \, J^2 - (\vec{p} \cdot \vec{J}\,)^2 + p^2 \, K^2 - (\vec{p} \cdot \vec{K}\,)^2 \\ &\quad + 2E \left[\vec{p} \cdot (\vec{J} \times \vec{K}) - i \, (\vec{p} \cdot \vec{K}\,)\right] \end{aligned} \tag{6.22}$$

Consider a special case of $\vec{p}$ along the z-direction, i.e. $p_x = p_y = 0$ and $p_z = p$. Then, we see that

$$
\begin{aligned}
-W^\mu(p)\, W_\mu(p) = w^2\, J^2 + p^2\,(J_x^2 + J_y^2 + K_x^2 + K_y^2) \\
+ 2E\,p\,(J_x\,K_y - J_y\,K_x - i\,K_z)
\end{aligned}
\tag{6.23}
$$

From (6.4), (6.6) and (6.7), we find

$$
-W^\mu(p)\, W_\mu(p) = 2 \begin{pmatrix} -p^2 & 0 & 0 & E\,p \\ 0 & w^2 & 0 & 0 \\ 0 & 0 & w^2 & 0 \\ -E\,p & 0 & 0 & E^2 \end{pmatrix}
\tag{6.24}
$$

Apply this operator to (6.18), to obtain

$$
\begin{aligned}
-W^\mu(p)\, W_\mu(p)\, e^\mu(\vec{p},0) &= 2w^2\, e^\mu(\vec{p},0) \\
-W^\mu(p)\, W_\mu(p)\, e^\mu(\vec{p},\pm 1) &= 2w^2\, e^\mu(\vec{p},\pm 1)
\end{aligned}
\tag{6.25}
$$

So, from considering a special case, we have proven a general formula

$$
-W^\mu(p)\, W_\mu(p) = j(j+1)\, w^2\, I = 2w^2\, I, \quad j = 1
\tag{6.26}
$$

where, again, I is the 4×4 identity matrix.

Consider now the spin operator $\vec{S}$. Once again we confine ourselves to $\vec{p}$ along the z-direction, i.e. $p_x = p_y = 0$ and $p_z = p$. Its z-component is

$$
\begin{aligned}
w\, S_z(p) &= E\, J_z - \frac{p^2}{E + w}\, J_z = w\, J_z, \quad \text{setting } p^2 = E^2 - w^2 \\
&= w \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{6.27}
$$

Applying it to (6.18), we find

$$
\begin{cases}
w\, S_z(p)\, e^\mu(\vec{p},0) = 0 \\
w\, S_z(p)\, e^\mu(\vec{p},\pm 1) = \pm w\, e^\mu(\vec{p},\pm 1)
\end{cases}
\tag{6.28}
$$

Next, we evaluate

$$w\,S_x(p) = E\,J_x + p\,K_y = i\begin{pmatrix} 0 & 0 & +p & 0 \\ 0 & 0 & 0 & 0 \\ +p & 0 & 0 & -E \\ 0 & 0 & +E & 0 \end{pmatrix}$$

$$w\,S_y(p) = E\,J_y - p\,K_x = i\begin{pmatrix} 0 & -p & 0 & 0 \\ -p & 0 & 0 & +E \\ 0 & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \end{pmatrix} \tag{6.29}$$

so that

$$w\,S_+(p) = \begin{pmatrix} 0 & p & +i\,p & 0 \\ p & 0 & 0 & -E \\ +i\,p & 0 & 0 & -i\,E \\ 0 & +E & +i\,E & 0 \end{pmatrix}, \quad w\,S_-(p) = \begin{pmatrix} 0 & -p & +i\,p & 0 \\ -p & 0 & 0 & +E \\ +i\,p & 0 & 0 & -i\,E \\ 0 & -E & +i\,E & 0 \end{pmatrix} \tag{6.30}$$

and obtain the desired result

$$\begin{cases} w\,S_\pm(p)\,e^\mu(\vec{p},0) = \sqrt{2}\,w\,e^\mu(\vec{p},\pm 1) \\ w\,S_\pm(p)\,e^\mu(\vec{p},\mp) = \sqrt{2}\,w\,e^\mu(\vec{p},0) \\ w\,S_\pm(p)\,e^\mu(\vec{p},\pm) = 0 \end{cases} \tag{6.31}$$

The formulas (6.27) and (6.30) represent concrete realization of the general operators $S_z(p)$ and $S_\pm(p)$ as 4×4 matrices in the case of $j=1$. The relationships (6.28) and (6.31) are in fact the same as (3.23) with $j=1$. It can be shown that

$$w^2\,S^2(p) = w^2\left[S_x^2(p) + S_y^2(p) + S_z^2(p)\right] = 2\begin{pmatrix} -p^2 & 0 & 0 & E\,p \\ 0 & w^2 & 0 & 0 \\ 0 & 0 & w^2 & 0 \\ -E\,p & 0 & 0 & E^2 \end{pmatrix} \tag{6.32}$$

$$= -W^\mu(p)\,W_\mu(p)$$

and, as before, we obtain

$$w^2\,S^2(p)\,e^\mu(\vec{p},m) = 2w^2\,e^\mu(\vec{p},m), \quad m = -1,0,+1 \tag{6.33}$$

appropriate for a $j = 1$ particle.

Next we work out the orbital-angular momentum operator $L(p)$ in the momentum space. Note

$$w\,\vec{L}(p) = w\,\vec{J} - w\,\vec{S}(p) \tag{6.34}$$

From (6.27), we see that

$$w\,L_z(p) = w \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0, \text{ i.e. a null operator} \tag{6.35}$$

and, from (6.29),

$$\begin{aligned} w\,L_x(p) &= i \begin{pmatrix} 0 & 0 & -p & 0 \\ 0 & 0 & 0 & 0 \\ -p & 0 & 0 & E-w \\ 0 & 0 & -(E-w) & 0 \end{pmatrix} \\ w\,L_y(p) &= i \begin{pmatrix} 0 & +p & 0 & 0 \\ +p & 0 & 0 & -(E-w) \\ 0 & 0 & 0 & 0 \\ 0 & E-w & 0 & 0 \end{pmatrix} \end{aligned} \tag{6.36}$$

Note

$$\begin{aligned} w^2\,L^2(p) &= w^2\left[L_x^2(p) + L_y^2(p) + L_z^2(p)\right] \\ &= 2 \begin{pmatrix} -p^2 & 0 & 0 & p\,(E-w) \\ 0 & -w(E-w) & 0 & 0 \\ 0 & 0 & -w(E-w) & 0 \\ -p\,(E-w) & 0 & 0 & (E-w)^2 \end{pmatrix} \end{aligned} \tag{6.37}$$

so that

$$\begin{cases} w^2\,L^2(p)\,e^\mu(\vec{p},0) = -2(E-w) \begin{pmatrix} p \\ 0 \\ 0 \\ E-w \end{pmatrix} \\ w^2\,L^2(p)\,e^\mu(\vec{p},\pm) = -2w(E-w)\,e^\mu(\vec{p},\pm) \end{cases} \tag{6.38}$$

We conclude that $L^2(p)$ is *not* a good quantum number for $e^\mu(\vec{p}, m)$. Only in the limit $p \to 0$, $L^2(p)$ is given by $\ell(\ell+1)$ with $\ell = 0$. The raising and lowering operators are

$$w\, L_+(p) = \begin{pmatrix} 0 & -p & -i\,p & 0 \\ -p & 0 & 0 & E-w \\ -i\,p & 0 & 0 & i\,(E-w) \\ 0 & -(E-w) & -i\,(E-w) & 0 \end{pmatrix}$$

$$w\, L_-(p) = \begin{pmatrix} 0 & +p & -i\,p & 0 \\ +p & 0 & 0 & -(E-w) \\ -i\,p & 0 & 0 & i\,(E-w) \\ 0 & E-w & -i\,(E-w) & 0 \end{pmatrix} \tag{6.39}$$

7 Spin-1 Wave Functions for Massless Particles

As a massless particle does not have a rest frame, we choose as its reference frame in which $p^\mu = (\omega, 0, 0, \omega)$. Then, from (6.20), we see that

$$W^0(p) = \omega\, J_z, \quad W_x(p) = \omega(J_x + K_y), \quad W_y(p) = \omega(J_y - K_x), \quad W_z(p) = \omega\, J_z \tag{7.1}$$

Define

$$A^\mu{}_\nu = (J_y - K_x)^\mu{}_\nu = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & +i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$B^\mu{}_\nu = (J_x + K_y)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & 0 \\ +i & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{pmatrix} \tag{7.2}$$

so that

$$-W^\mu(p)\, W_\mu(p) = \omega^2 \left[A^2 + B^2\right] = 2 \begin{pmatrix} -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 \end{pmatrix} \tag{7.3}$$

The commutation relations [note $A\,B = B\,A = 0$]

$$[A, B] = 0, \quad [J_z, A] = +i\,B, \quad [J_z, B] = -i\,A \tag{7.4}$$

show that a set $\{J_z, A, B\}$ constitutes a Lie algebra. Introduce an Hilbert space in which J_z, A and B are represented by Hermitian operators. [Note that matrices (7.2) are *not* Hermitian.] In this space, we introduce spin-1 massless states $|\vec{p}, \lambda\rangle$ via

$$\begin{aligned} J_z\,|\vec{p}, \lambda\rangle &= \lambda\,|\vec{p}, \lambda\rangle \\ A\,|\vec{p}, \lambda\rangle &= B\,|\vec{p}, \lambda\rangle = 0; \quad \text{(from experiment)} \end{aligned} \tag{7.5}$$

where the z-axis has been chosen along $\vec{p}$. So $\lambda = \pm 1$ are the spin-1 helicities. On this basis, we have $W^\mu(p)\,W_\mu(p) = 0$, which agrees with (6.26) in the limit $w \to 0$. It is *very* interesting to note (a coincidence?) that $e^\mu(\vec{p}, \pm 1)$ defined in (6.18) as a column vector satisfies the condition $W^\mu(p)\,W_\mu(p) = 0$ [see (7.3)]. Indeed, it should be noted that, using the 4×4 matrices defined in (7.2), we obtain

$$A\,e(\vec{p}, \pm 1) = \pm \frac{i}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad B\,e(\vec{p}, \pm 1) = + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{7.6}$$

which are clearly in conflict with (7.5).

8 Composite Wave Functions in Momentum Space

Consider a composite system consisting of two spin-1 particles with momenta $\vec{p}_1$ and $\vec{p}_2$. The spin-S wave functions (more precisely, the *total intrinsic spin* is S—see later) for the composite system are given by

$$e^{\mu\nu}(\vec{p}, S m_s) = \sum_{m_1\, m_2} (1 m_1 \; 1 m_2 | S m_s)\, e^\mu(\vec{p}_1, m_1)\, e^\nu(\vec{p}_2, m_2) \tag{8.1}$$

where $\vec{p} = \vec{p}_1 + \vec{p}_2$ and $S = 0, 1, 2$. Consider the wave functions at rest. In this case, we set $\vec{p} = 0$, $\vec{p}_1 = \vec{q}$ and $\vec{p}_2 = -\vec{q}$ and write

$$e^{\mu\nu}(\overset{\circ}{\Omega}, S m_s) = \sum_{m_1\, m_2} (1 m_1 \; 1 m_2 | S m_s)\, e^\mu(\vec{q}, m_1)\, e^\nu(-\vec{q}, m_2) \tag{8.2}$$

Note that the summation above is in reality over one variable only—m_1, since $m_2 = m - m_1$. Apply a rotation $R^S(\Omega)$ to find

$$\begin{aligned}
R^S\, e^{\mu\nu}(\overset{\circ}{\Omega}, Sm_s) &= \exp[-i\,\vec{\theta}\cdot\vec{S}(p)]\, e^{\mu\nu}(\overset{\circ}{\Omega}, Sm_s) \\
&= \sum_{m_1\, m_2} (1m_1\ 1m_2|Sm_s)\, \Big\{\exp[-i\,\vec{\theta}\cdot\vec{S}^{(1)}(\vec{q})]\, e^{\mu}(\vec{q}, m_1)\Big\}\, \Big\{\exp[-i\,\vec{\theta}\cdot\vec{S}^{(2)}(-\vec{q})]\, e^{\nu}(-\vec{q}, m_2)\Big\} \\
&= \sum_{\substack{m_1\, m_2\\ m_1'\, m_2'}} (1m_1\ 1m_2|Sm_s)\, \Big\{e^{\mu}(\vec{q}, m_1')\, D^{(1)}_{m_1'\, m_1}(R)\Big\}\, \Big\{e^{\nu}(-\vec{q}, m_2')\, D^{(1)}_{m_2'\, m_2}(R)\Big\}
\end{aligned} \tag{8.3}$$

where $\vec{S}(p) = \vec{S}^{(1)}(\vec{q}) + \vec{S}^{(2)}(-\vec{q})$. But

$$\begin{aligned}
D^{(1)}_{m_1'\, m_1}(R)\, D^{(1)}_{m_2'\, m_2}(R) &= \sum_{j\, m'\, n} (1m_1'\ 1m_2'|j\ m_s')\, (1m_1\ 1m_2|j\ n)\, D^{j}_{m_s'\, n}(R) \\
\sum_{m_1\, m_2} (1m_1\ 1m_2|Sm_s)\, (1m_1\ 1m_2|j\ n) &= \delta_{Sj}\, \delta_{m_s\, n}
\end{aligned} \tag{8.4}$$

so that

$$\begin{aligned}
R^S\, e^{\mu\nu}(\overset{\circ}{\Omega}, Sm_s) &= \sum_{m_1'\, m_2'\, m'} (1m_1'\ 1m_2'|Sm_s')\, e^{\mu}(\vec{q}, m_1')\, e^{\nu}(-\vec{q}, m_2')\, D^{(2)}_{m'\, m}(R) \\
&= \sum_{m'} e^{\mu\nu}(\overset{\circ}{\Omega}, Sm_s')\, D^{S}_{m_s'\, m_s}(R)
\end{aligned} \tag{8.5}$$

Here the summation on m_2' is superfluous, because $m_2' = m' - m_1'$. Note that the formula above is the momentum-space analogue of the (4.12); it shows that spin-S is the total intrinsic spin built out of two spin-1 states.

In a paper[3] on covariant helicity-coupling amplitudes, it was shown that a system consisting of the vectors aligned along the z-axis ($\hat{k}$) plays an important role. For illustration, we set $\overset{\circ}{\Omega}(\theta, \phi)$ to $\Omega_0 = \Omega(0, 0)$ and write (8.5) again

$$\begin{aligned}
R^S\, e^{\mu\nu}(\Omega_0, S\rho) &= \sum_{\lambda\, \delta} (1\lambda\ 1\, {-\nu}|S\delta)\, e^{\mu}(q\hat{k}, \lambda)\, e^{\nu}(-q\hat{k}, -\nu)\, D^{S}_{\delta\, \rho}(R) \\
&= \sum_{\delta} e^{\mu\nu}(\Omega_0, S\delta)\, D^{S}_{\delta\, \rho}(R)
\end{aligned} \tag{8.6}$$

where λ and ν are the spin-1 helicities and ρ and δ are the spin-S helicities.

A Proof of the Relation (2.34)

Start with a set of four-vectors a, b, c and d. One can form a Lorentz-invariant

$$(abcd) = \varepsilon^{\rho\sigma\tau\upsilon}\, a_\rho\, b_\sigma\, c_\tau\, d_\upsilon \tag{A.1}$$

Under a proper homogeneous Lorentz transformation Λ, the four-vectors transform into

$$a'_\alpha = \Lambda_\alpha{}^\rho\, a_\rho, \quad b'_\beta = \Lambda_\beta{}^\sigma\, b_\sigma, \quad c'_\gamma = \Lambda_\gamma{}^\tau\, c_\tau, \quad d'_\delta = \Lambda_\delta{}^\upsilon\, d_\upsilon \tag{A.2}$$

so that

$$(a'b'c'd') = \varepsilon^{\alpha\beta\gamma\delta}\, a'_\alpha\, b'_\beta\, c'_\gamma\, d'_\delta \tag{A.3}$$

Since a, b, c and d are arbitrary and $(a'b'c'd') = (abcd)$, one can conclude

$$\begin{aligned} \varepsilon^{\rho\sigma\tau\upsilon} &= \varepsilon^{\alpha\beta\gamma\delta}\, \Lambda_\alpha{}^\rho\, \Lambda_\beta{}^\sigma\, \Lambda_\gamma{}^\tau\, \Lambda_\delta{}^\upsilon \\ \varepsilon_{\rho\sigma\tau\upsilon} &= \varepsilon_{\alpha\beta\gamma\delta}\, \Lambda^\alpha{}_\rho\, \Lambda^\beta{}_\sigma\, \Lambda^\gamma{}_\tau\, \Lambda^\delta{}_\upsilon \end{aligned} \tag{A.4}$$

For a set $\{\rho\,\sigma\,\tau\,\upsilon\} = \{0, 1, 2, 3\}$, $\varepsilon^{0123} = -1$ and $\varepsilon_{0123} = +1$, the second equation above reduces to

$$1 = \begin{vmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{vmatrix} \tag{A.5}$$

It is clear that an arbitrary set $\{\rho\,\sigma\,\tau\,\upsilon\}$ is merely a permutation of the columns in the determinant above; even (odd) permutations result in $+1(-1)$ on the right, as expected. So (A.4) and (A.5) are equivalent.

Any proper homogeneous Lorentz transformation is a product of a rotation and a boost

$$\Lambda^\mu{}_\nu(\vec{v}/c) = \Lambda^\mu{}_\rho[R_z(\alpha)]\, \Lambda^\rho{}_\sigma[R_y(\beta)]\, \Lambda^\sigma{}_\upsilon[R_z(\gamma)]\, \Lambda^\upsilon{}_\nu[L_z(v/c)] \tag{A.6}$$

where $\vec{v}$ is the velocity of the Lorentz transformation. The expression above corresponds to a boost along the z-axis, followed by a rotation involving the the Euler angles (α, β, γ) through '*successive*' coordinate transformations. The transformations are familiar

$$\Lambda^\mu{}_\nu[R_z(\alpha)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda^\mu{}_\nu[R_y(\beta)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & 0 & \sin\beta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\beta & 0 & \cos\beta \end{pmatrix} \tag{A.7}$$

and

$$\Lambda^\mu{}_\nu[R_z(\gamma)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma & 0 \\ 0 & \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \Lambda^\mu{}_\nu[L_z(v/c)] = \begin{pmatrix} \cosh\xi & 0 & 0 & \sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix} \quad \text{(A.8)}$$

where $v/c = \tanh\xi$. It is clear that each of the determinants above is equal to unity. The equality of (A.5) follows from the property $\det(AB) = \det(A)\,\det(B)$.

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