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Two-channel Mass Matrix
—Version I—

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abstract

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1 Introduction

2 Eigenvalue Problem

Consider a two-by-two mixing matrix $\mathcal{M}$, where the elements are all real,

$$\begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix} = \mathcal{M} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \quad \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix} = \begin{pmatrix} m_a^2 & -\epsilon \\ -\epsilon & m_b^2 \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \tag{2.1}$$

Let U be a two-by-two unitary matrix (real, in our case) which diagonalizes the mass matrix given above

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2.2}$$

Then

$$U \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix} = U \begin{pmatrix} m_a^2 & -\epsilon \\ -\epsilon & m_b^2 \end{pmatrix} U^{-1} U \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \tag{2.3}$$

and

$$\mathcal{M}' = U\,\mathcal{M}\,U^{-1}, \quad \mathcal{M}' = U \begin{pmatrix} m_a^2 & -\epsilon \\ -\epsilon & m_b^2 \end{pmatrix} U^{-1} = \begin{pmatrix} \mathcal{M}'_{11} & \mathcal{M}'_{12} \\ \mathcal{M}'_{21} & \mathcal{M}'_{22} \end{pmatrix} \tag{2.4}$$

where

$$\begin{aligned} \mathcal{M}'_{11} &= m_a^2 \cos^2\theta + m_b^2 \sin^2\theta + 2\epsilon \sin\theta \cos\theta \\ \mathcal{M}'_{22} &= m_a^2 \sin^2\theta + m_b^2 \cos^2\theta - 2\epsilon \sin\theta \cos\theta \\ \mathcal{M}'_{12} &= \mathcal{M}'_{21} \\ &= (m_a^2 - m_b^2) \sin\theta \cos\theta - \epsilon (\cos^2\theta - \sin^2\theta) \\ &= (m_a^2 - m_b^2) (\sin 2\theta)/2 - \epsilon (\cos 2\theta) \end{aligned} \tag{2.5}$$

So $\mathcal{M}'$ is diagonalized, if we set

$$\tan 2\theta = \frac{2\epsilon}{m_a^2 - m_b^2} \tag{2.6}$$

We shall introduce the following recombination of masses:

$$\begin{aligned} \sigma &= \frac{1}{2}(m_a^2 + m_b^2) \\ \delta &= \frac{1}{2}(m_a^2 - m_b^2) \end{aligned} \tag{2.7}$$

so that $m_a^2 = \sigma + \delta$ and $m_b^2 = \sigma - \delta$. Then we see that

$$\begin{aligned} \mathcal{M}'_{11} &= \sigma + \delta \cos 2\theta + \epsilon \sin 2\theta \\ \mathcal{M}'_{22} &= \sigma - \delta \cos 2\theta - \epsilon \sin 2\theta \end{aligned} \tag{2.8}$$

Note

$$\tan 2\theta = \epsilon/\delta, \quad \cos 2\theta = \frac{\delta}{\sqrt{\delta^2+\epsilon^2}} \quad \text{and} \quad \sin 2\theta = \frac{\epsilon}{\sqrt{\delta^2+\epsilon^2}} \tag{2.9}$$

so that

$$\begin{cases} \mathcal{M}'_{11} = \sigma + \sqrt{\delta^2+\epsilon^2} \\ \mathcal{M}'_{22} = \sigma - \sqrt{\delta^2+\epsilon^2} \end{cases} \tag{2.10}$$

Define

$$\begin{aligned} &\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = U \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = U \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \\ &\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \mathcal{M}' \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \end{aligned} \tag{2.11}$$

where

$$\begin{cases} m_1^2 = \sigma + \sqrt{\delta^2+\epsilon^2} \\ m_2^2 = \sigma - \sqrt{\delta^2+\epsilon^2} \end{cases} \tag{2.12}$$

We find that the mass shift from the average value is given by a quadratic sum of δ, the difference in mass^2 divided by two, and the mixing parameter ϵ. The formula above leads the center-of-gravity formula

$$m_1^2 + m_2^2 = m_a^2 + m_b^2 \tag{2.13}$$

and the mixing angle in terms of the mass eigenvalues

$$\tan^2 2\theta = \frac{4\epsilon^2}{4\delta^2} = \frac{(m_1^2 - m_2^2)^2 - 4\delta^2}{4\delta^2} = \left(\frac{m_1^2 - m_2^2}{m_a^2 - m_b^2} \right)^2 - 1 \tag{2.14}$$

In addition, we see that there would be no mass shift, if $\epsilon = 0$, i.e. $M_1 = m_a$ and $M_2 = m_b$. If $\delta = 0$, then we may set $m = m_a = m_b$ and obtain

$$\begin{cases} m_1^2 = m^2 + \epsilon \\ m_2^2 = m^2 - \epsilon \end{cases} \tag{2.15}$$

and, since $\theta = 45°$ from (2.6),

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad U^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{2.16}$$

3 Decay Amplitudes

Consider the decay of the $a_1(1260)$ and $b_1(1235)$ into ρ and π octets. The C invariance implies that the f_1 cannot couple to $\omega\,\eta_8$ or to $\phi\,\eta_8$, while the h_1 should decay into $\omega\,\eta_8$ or $\phi\,\eta_8$. Inspection of the $SU(3)$ isoscalar table for $\mathbf{8} \to \mathbf{8} \otimes \mathbf{8}$ reveals that the symmetric octet $\mathbf{8}_1$ couples to $\omega\,\eta_8$ or $\phi\,\eta_8$ but the antisymmetric octet $\mathbf{8}_2$ does not (see Tables I and II).

Table I: 8_1 Isoscalars [for $b_1(1235)$]

Y	I	Wave Function
1	1/2	$\frac{3}{2\sqrt{5}}\,(K\,\rho - \pi\,K^*) - \frac{1}{2\sqrt{5}}\,(K\,\omega_8 + \eta_8\,K^*)$
0	1	$-\sqrt{\frac{3}{10}}\,(K\,\bar{K}^* + \bar{K}\,K^*) + \frac{1}{\sqrt{5}}(\pi\,\omega_8 + \eta_8\,\rho)$
0	0	$\frac{1}{\sqrt{10}}\,(K\,\bar{K}^* - \bar{K}\,K^*) - \sqrt{\frac{3}{5}}\,\pi\,\rho - \sqrt{\frac{1}{5}}\,\eta_8\,\omega_8$

Table II: 8_2 Isoscalars [for $a_1(1260)$]

Y	I	Wave Function
1	1/2	$\frac{1}{2}\,(K\,\rho + \pi\,K^* + K\,\omega_8 - \eta_8\,K^*)$
0	1	$\frac{1}{\sqrt{6}}\,(K\,\bar{K}^* - \bar{K}\,K^*) + \sqrt{\frac{2}{3}}\,\pi\,\rho$
0	0	$\frac{1}{\sqrt{2}}\,(K\,\bar{K}^* + \bar{K}\,K^*)$

Let K_a and K_b be the strange partners of the $a_1(1260)$ and $b_1(1235)$. Denoting the decay coupling constants by f_a and f_b, respectively, we see that

$$\begin{aligned} \langle K\,\rho\,|K_a\rangle &= \frac{1}{2}\,f_a, & \langle \pi\,K^*|K_a\rangle &= \frac{1}{2}\,f_a \\ \langle K\,\rho\,|K_b\rangle &= \frac{3}{2\sqrt{5}}\,f_b, & \langle \pi\,K^*|K_b\rangle &= -\frac{3}{2\sqrt{5}}\,f_b \end{aligned} \tag{3.1}$$

Using (2.11), we set

$$\begin{aligned} |K_1\rangle &\equiv |K_1(1270)\rangle = \cos\theta\,|K_a\rangle - \sin\theta\,|K_b\rangle \\ |K_2\rangle &\equiv |K_1(1400)\rangle = \sin\theta\,|K_a\rangle + \cos\theta\,|K_b\rangle \end{aligned} \tag{3.2}$$

so that

$$\begin{aligned}
\langle K\,\rho\,|K_1\rangle &= \frac{1}{2}\,f_a\,\cos\theta - \frac{3}{2\sqrt{5}}\,f_b\,\sin\theta \\
\langle \pi\,K^*|K_1\rangle &= \frac{1}{2}\,f_a\,\cos\theta + \frac{3}{2\sqrt{5}}\,f_b\,\sin\theta \\
\langle K\,\rho\,|K_2\rangle &= \frac{1}{2}\,f_a\,\sin\theta + \frac{3}{2\sqrt{5}}\,f_b\,\cos\theta \\
\langle \pi\,K^*|K_2\rangle &= \frac{1}{2}\,f_a\,\sin\theta - \frac{3}{2\sqrt{5}}\,f_b\,\cos\theta
\end{aligned} \tag{3.3}$$

Form the appropriate ratios

$$\begin{aligned}
\left|\frac{\langle \pi\,K^*|K_1\rangle}{\langle K\,\rho\,|K_1\rangle}\right| &= \left|\frac{f_a\,\cos\theta + \frac{3}{\sqrt{5}}\,f_b\,\sin\theta}{f_a\,\cos\theta - \frac{3}{\sqrt{5}}\,f_b\,\sin\theta}\right| \\
\left|\frac{\langle K\,\rho\,|K_2\rangle}{\langle \pi\,K^*|K_2\rangle}\right| &= \left|\frac{\frac{\sqrt{5}}{3}\,\sin\theta + f_b\,\cos\theta}{\frac{\sqrt{5}}{3}\,\sin\theta - f_b\,\cos\theta}\right|
\end{aligned} \tag{3.4}$$

In order to conform to the notations of Suzuki[1], we define

$$\theta = \frac{\pi}{2} - \theta_{_K}, \quad x = -\frac{f_b}{f_a} \tag{3.5}$$

to find

$$\begin{aligned}
\left|\frac{\langle \pi\,K^*|K_1\rangle}{\langle K\,\rho\,|K_1\rangle}\right|_{K_1(1270)} &= \left|\frac{\tan\theta_{_K} - \frac{3}{\sqrt{5}}\,x}{\tan\theta_{_K} + \frac{3}{\sqrt{5}}\,x}\right| = 0.21 \\
\left|\frac{\langle K\,\rho\,|K_2\rangle}{\langle \pi\,K^*|K_2\rangle}\right|_{K_1(1400)} &= \left|\frac{\frac{\sqrt{5}}{3} - x\,\tan\theta_{_K}}{\frac{\sqrt{5}}{3} + x\,\tan\theta_{_K}}\right| = 0.21
\end{aligned} \tag{3.6}$$

where we have taken only the s-wave portion of the decay modes and corrected for the phase-space factor. The solutions are

$$\begin{aligned}
&(a) \quad \theta_{_K} = 33^\circ, \qquad x = 0.75 \\
&(b) \quad \theta_{_K} = 45^\circ, \qquad x = 1.14 \\
&(c) \quad \theta_{_K} = 57^\circ, \qquad x = 0.74 \\
&(d) \quad \theta_{_K} = 45^\circ, \qquad x = 0.48
\end{aligned} \tag{3.7}$$

4 Mass Formulas

For an ideally mixed nonet of mesons, we may use the static quark model to obtain, e.g. for the vector mesons (and also for the tensor mesons),

$$\begin{aligned}
m^2(\rho) + m^2(\phi) &= 2\,m^2(K^*) \\
m^2(\rho) &= m^2(\omega)
\end{aligned} \tag{4.1}$$

A heuristic approach to understanding these formulas is examine them in their linear form in mass. Substitute $m(\rho) = m(\omega) = 2\,m_n$, $m(\phi) = 2\,m_s$ and $m(K^*) = m_s + m_n$, where m_n is the average of the 'up' and 'down' quark mass and m_s is the strange quark mass. We see readily that the formulas are satisfied in their linear form in mass. We deduce the masses for K_a and K_b in the absence of mixing, from (4.1),

$$\begin{aligned} m_a = m(K_a) &= \sqrt{(1260)^2 + (1415)^2)/2} = 1340 \text{ MeV} \\ m_b = m(K_b) &= \sqrt{(1232)^2 + (1380)^2)/2} = 1308 \text{ MeV} \end{aligned} \tag{4.2}$$

Applying these values to (2.14), we find $\theta_{_K} = 32°$ or $\theta_{_K} = 58°$. So the solutions (a) and (c) are found to be compatible with these.

References

[1] M. Suzuki, Phys. Rev. D**47**, 1252 (1993).

[2] J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963).