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On $SU(3)$ Representations of $q\bar{q}+q\bar{q}$ mesons —Version I—

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abstract

Some general results are given on the product representation of two irreducible representations of $SU(3)$. These results are applied to a p-wave state coupling to two ground-state pseudoscalars.

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1 Introduction

The purpose of this note is to give (in section 2) some technical material concerning a set of irreducible representations resulting from the product of two irreducible representations of $SU(3)$. It is then used in section 3 to show that charge-conjugation operations play a fundamentally different role depending on certain characteristics of the irreducible representations under study.

Conclusions are given in Section 3.

2 Irreducible Representations

Let $D(p,q)$ be an irreducible representation characterized by two integers p and q. For a physically realizable representation, one must have $p-q=3\,n$, where $n=0,\pm1,\pm2,\pm3,\ldots$. The number of basis vectors in an irreducible representation is given by the dimensionality N of the representation[1]

$$N=(1+p)\,(1+q)\,\left[1+\frac{1}{2}\,(p+q)\right] \tag{1}$$

There are two Casimir operators F^2 and G^3 with the eigenvalues f^2 and g^3. They are given by[2]

$$\begin{aligned} f^2 &= \frac{1}{3}\,\left[p^2+q^2+p\,q+3\,(p+q)\right] \\ g^3 &= \frac{1}{18}\,(p-q)\,(2p+q+3)\,(2q+p+3) \end{aligned} \tag{2}$$

So an irreducible representation can be equivalently characterized by $D(f^2,g^3)$ corresponding to the two Casimir eigenvalues. See Table I for a few examples of practical importance.

Table I: Irreducible Representations of $SU(3)$

N	1	8	10	$\overline{10}$	27
(p,q)	$(0,0)$	$(1,1)$	$(3,0)$	$(0,3)$	$(2,2)$
(f^2,g^3)	$(0,0)$	$(3,0)$	$(6,9)$	$(6,-9)$	$(8,0)$

An eigenstate (or a wave function) belonging to an irreducible representation is given by the eigenvalues corresponding to a set of five commuting operators

$$\{F^2,\ G^3,\ I^2,\ I_3,\ Y\} \tag{3}$$

where I is the isotopic spin and Y is the hypercharge. It is conventional to use I and Y for both operators and eigenvalues. Thus, the eigenvalue for the $SU(2)$ Casimir operator I^2 is $I(I+1)$, and that for Y is just $Y = B + S$, but the eigenvalue for I_3 is denoted m here. Introduce new notations for convenience:

$$\mu = \{f^2, g^3\}, \qquad \sigma = \{I,\ Y\}, \qquad \text{and} \qquad \nu = \{I,\ m,\ Y\} \tag{4}$$

Then, the eigenstate can be given a compact notation $\phi^{(\mu)}_{\nu}$.

Consider a product representation $D(p_1, q_1) \otimes D(p_2, q_2)$. It can be expanded as a direct sum of irreducible representations. The eigenstates of each irreducible representation in the expansion are given by

$$\psi \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ & & \nu \end{pmatrix} = \sum_{\nu_1,\nu_2} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \phi^{(\mu_1)}_{\nu_1} \phi^{(\mu_2)}_{\nu_2} \tag{5}$$

following the notations used previously[1]. The subscript γ is a label which distinguishes two irreducible representations with the same (p, q) or (f^2, g^3), e.g. $\mathbf{8_1}$ and $\mathbf{8_2}$ which appear in (7). The transformation matrix in (5) is real and orthogonal—given by

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} = \left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu_\gamma \\ \sigma_1 & \sigma_2 & \sigma \end{array}\right) (I_1 m_1\, I_2 m_2 | I m) \tag{6}$$

where the first element on the right-hand side is the $SU(3)$ isoscalar factor and the second element is the usual $SU(2)$ Clebsch-Gordan coefficient.

3 Decay Process $X \to a_1 + a_2$

Let X be a non-strange p-wave resonance decaying into two ground-state pseudoscalars a_1 and a_2, i.e. a_1 and a_2 come from the set $\{\pi,\ K,\ \bar{K}$ or $\eta\}$. (Here we assume that η is a pure octet member and η' is a pure $SU(3)$ singlet.) The following expansion gives relevant irreducible representations

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8_1} \oplus \mathbf{8_2} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27} \tag{7}$$

Assume now that $SU(3)$ is strictly conserved in the decay. Then the decay Hamiltonian $\mathcal{M}$ commutes with the Casimir operators of $SU(3)$

$$[F^2, \mathcal{M}] = [G^3, \mathcal{M}] = 0 \tag{8}$$

If X belongs to a representation μ_γ, then the wave function for the decay product must also belong to μ_γ, and the resulting decay amplitude A can be written

$$A_\nu^{(\mu_\gamma)} = \langle \psi_\nu^{(\mu_\gamma)}(a_1 a_2) | \mathcal{M} | \phi_\nu^{(\mu_\gamma)}(X) \rangle \tag{9}$$

where $\psi_\nu^{(\mu_\gamma)}$ is a shorthand notation for the wave function given in (5) in which labels μ_1 and μ_2 have been suppressed. Of course, $\mathcal{M}$ commutes in addition with I^2, I_3 and Y and, in particular, with the charge-conjugation operator (to be covered later in more detail).

Bose symmetrization requires that only the antisymmetric wave functions in (7) are relevant for this decay process, and they are $\mathbf{8_2}$, $\mathbf{10}$ and $\overline{\mathbf{10}}$. Since we are dealing with a p-wave state, it is clear that all the states in the representations belong to $J^P = 1^-$. We shall use the notations in Tables II and III for identifying individual states.

Table II: Names for Multiplet $\mathbf{8_2}$

Multiplet	Y	I	m	Q	Name
$\mathbf{8_2}$	$+1$	$1/2$	$-1/2$	-1	$\bar{K}^{*0}$
			$+1/2$	0	$\bar{K}^{*+}$
$\mathbf{8_2}$	0	1	-1	-1	ρ^-
			0	0	ρ^0
			$+1$	$+1$	ρ^+
$\mathbf{8_2}$	0	0	0	0	ω
$\mathbf{8_2}$	-1	$1/2$	$-1/2$	-1	K^{*-}
			$+1/2$	0	$\bar{K}^{*0}$

Table III: Names for Multiplet 20 ($\mathbf{10} \oplus \overline{\mathbf{10}}$)					
Multiplet	Y	I	m	Q	Name
$\overline{\mathbf{10}}$	$+2$	0	0	$+1$	$\bar{\omega}^+$
$\mathbf{10}$	$+1$	3/2	+3/2	$+2$	δ^{++}
			+1/2	$+1$	δ^+
			-1/2	0	δ^0
			-3/2	-1	δ^-
$\overline{\mathbf{10}}$	$+1$	1/2	+1/2	$+1$	$\bar{K}^{*+}$
			-1/2	0	$\bar{K}^{*0}$
$\mathbf{10}$ and $\overline{\mathbf{10}}$	0	1	+1	$+1$	σ^+ and $\bar{\sigma}^+$
			0	0	σ^0 and $\bar{\sigma}^0$
			-1	-1	σ^- and $\bar{\sigma}^-$
$\mathbf{10}$	-1	1/2	+1/2	0	K^{*0}
			-1/2	-1	K^{*-}
$\overline{\mathbf{10}}$	-1	3/2	+3/2	$+1$	$\bar{\delta}^+$
			+1/2	0	$\bar{\delta}^0$
			-1/2	-1	$\bar{\delta}^-$
			-3/2	-2	$\bar{\delta}^{--}$
$\mathbf{10}$	-2	0	0	-1	ω^-

The charge-conjugation property of an $SU(3)$ family can be studied by examining the wave function for a non-strange neutral member. Let $\nu_0 = \{0, 0, 0\}$ and $\nu_3 = \{1, 0, 0\}$. One finds readily

$$
\begin{aligned}
\omega = \psi \begin{pmatrix} 8 & 8 & 8_2 \\ & & \nu_0 \end{pmatrix} &= \frac{1}{2}\left(\bar{K}^0\, K^0 - K^0\, \bar{K}^0\right) - \frac{1}{2}\left(K^-\, K^+ - K^+\, K^-\right) \\
\rho^0 = \psi \begin{pmatrix} 8 & 8 & 8_2 \\ & & \nu_3 \end{pmatrix} &= \sqrt{\frac{1}{3}}\,\left(\pi^+\, \pi^- - \pi^-\, \pi^+\right) - \sqrt{\frac{1}{12}}\left(\bar{K}^0\, K^0 - K^0\, \bar{K}^0\right) \\
&\quad - \sqrt{\frac{1}{12}}\left(K^-\, K^+ - K^+\, K^-\right)
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
\sigma^0 = \psi \begin{pmatrix} 8 & 8 & 10 \\ & & \nu_3 \end{pmatrix} &= \sqrt{\frac{1}{12}}\left(\pi^+\, \pi^- - \pi^-\, \pi^+\right) + \sqrt{\frac{1}{12}}\left(\bar{K}^0\, K^0 - K^0\, \bar{K}^0\right) \\
&\quad + \sqrt{\frac{1}{12}}\left(K^-\, K^+ - K^+\, K^-\right) + \frac{1}{2}\,\left(\pi^0\eta - \eta\pi^0\right) \\
\bar{\sigma}^0 = \psi \begin{pmatrix} 8 & 8 & \overline{10} \\ & & \nu_3 \end{pmatrix} &= -\sqrt{\frac{1}{12}}\left(\pi^+\, \pi^- - \pi^-\, \pi^+\right) - \sqrt{\frac{1}{12}}\left(\bar{K}^0\, K^0 - K^0\, \bar{K}^0\right) \\
&\quad - \sqrt{\frac{1}{12}}\left(K^-\, K^+ - K^+\, K^-\right) + \frac{1}{2}\,\left(\pi^0\eta - \eta\pi^0\right)
\end{aligned} \tag{11}
$$

In order to proceed further, one needs a precise, self-consistent definition of the charge conjugation C. Let labels k and $\bar{k}$ stand for particles and anti-particles. One may define[3]

$$
C\,|k\;I\,m\rangle = g_c\,(-)^{I+m}\,|\bar{k}\;I\;-m\rangle, \qquad C\,|\bar{k}\;I\,m\rangle = \bar{g}_c\,(-)^{I+m}\,|k\;I\;-m\rangle \tag{12}
$$

where

$$
g_c = \xi\,(-)^{Y/2+I}, \qquad \bar{g}_c = \xi\,(-)^{\overline{Y}/2+I} \tag{13}
$$

Here $Y = \pm 1$, $\overline{Y} = -Y$, and

$$
Q = Y/2 + m \tag{14}
$$

ξ is simply the charge conjugation of the non-strange, neutral members of the $SU(3)$ family, irrespective of their isotopic spin. This holds true for $\mathbf{8_2}$; the two formulas of (10) show that both have $C = -1$ so that $\mathbf{8_2}$ belongs to the $J^{PC} = 1^{--}$ family. $\xi = -1$ in this case.

But the situation with $\mathbf{10}$ and $\overline{\mathbf{10}}$ is quite different. The wave functions (11) reveal that the non-strange isovector members consist of a mixture of $C = +1$ and $C = -1$. Since the decay Hamiltonian $\mathcal{M}$ commutes with the C operator, the couplings to wrong-C final states can be shown to be zero, by observing

$$
\langle \psi'(C = +1) + \psi'(C = -1)|\mathcal{M}|X\rangle \tag{15}
$$

where $\psi \equiv \psi'(C=+1) + \psi'(C=-1)$. One inserts $C^\dagger C = I$ in the middle of the formula above and propagates the operators towards right and left. An X with $J^{PC} = 1^{-+}$ couples to $\pi\eta$ only, while an X with $J^{PC} = 1^{--}$ decays into $\pi\pi$ and $K\bar{K}$. But the decay strengths are somewhat reduced because of the $SU(3)$ isoscalar factors.

Consider now the strange members of **10** and $\overline{\mathbf{10}}$. There are seven in **10** and another seven in $\overline{\mathbf{10}}$. But the C operation takes one in **10** to another in $\overline{\mathbf{10}}$, so that the masses must be exactly the same for the charge-conjugate pair. There are thus a total of 14 strange states, grouped into three sets of masses for $I = 3/2$, $I = 1/2$ and $I = 0$.

There are a total of 20 different states in **10** and $\overline{\mathbf{10}}$ representations. In general, a state X with $J^P = 1^-$ belonging to **10** can decay into $a_1 + a_2$, while its anti-particle partner $\overline{X}$ decays into $\bar{a}_1 + \bar{a}_2$ with exactly the same strength. Only in the non-strange $I = 1$ sector, the X comes in eigenstates of the G-parity, i.e. $J^{PC} = 1^{-+}$ and $J^{PC} = 1^{--}$. Let $\nu_3 = \{1, m, 0\}$. Then, the normalized wave functions $\psi_\pm$ for $C = \pm$ are given by

$$\sqrt{2}\begin{pmatrix} \pi_1^m \\ \rho^m \end{pmatrix} = \sqrt{2}\,\psi_\pm \begin{pmatrix} 8 & 8 & 10 \\ & & \nu_3 \end{pmatrix} = \psi \begin{pmatrix} 8 & 8 & 10 \\ & & \nu_3 \end{pmatrix} \pm \psi \begin{pmatrix} 8 & 8 & \overline{10} \\ & & \nu_3 \end{pmatrix} = \sigma^m \pm \bar{\sigma}^m \tag{16}$$

for each $m = -1, 0, +1$. Let $\mathcal{H}$ be the Hamiltonian that gives rise to the masses in the limit of exact $SU(3)$. Then , one sees that

$$\mathcal{H}\,|\psi_\pm\rangle = M_\pm\,|\psi_\pm\rangle \tag{17}$$

where $\psi_\pm$ is now a wave function extended to arbitrary ν and $M_\pm$ is the corresponding putative mass. The $SU(3)$ Casimir operators act on the wave functions according to

$$F^2\,|\psi_\pm\rangle = 6\,|\psi_\pm\rangle \qquad \text{and} \qquad G^3\,|\psi_\pm\rangle = 9\,|\psi_\mp\rangle \tag{18}$$

which lead to, because of G^3,

$$\mathcal{H}\,|\psi_\mp\rangle = M_\pm\,|\psi_\mp\rangle \tag{19}$$

For this to be consistent with (17), one must have $M_+ = M_-$.

In conclusion, the multiplet 20 with $J^P = 1^-$ derived from **10** and $\overline{\mathbf{10}}$ have seven strange particles (δ, K^* and ω^-) and their anti-particles ($\bar{\delta}$, $\bar{K}^*$ and $\bar{\omega}^+$) plus a non-strange isovector with $J^{PC} = 1^{-+}$ (π_1) decaying into $\pi\eta$ and another one with $J^{PC} = 1^{--}$ (ρ) and *equal* mass decaying into $\pi\pi$ and $K\bar{K}$. An important caveat is that these states can mix with others of the same J^{PC}. The $I^G(J^{PC}) = 1^+(1^{--})$ (ρ) in this multiplet 20 can mix with its counterpart in $\mathbf{8_2}$, for example, and also with the $\rho(1450)$ and $\rho(1700)$ listed in the PDG[4]. From this point of view, the $\rho(1450)$ could be a complicated mixture of a radially excited quarkonium, a ($q\bar{q}$ + gluon) hybrid and a ($q\bar{q} + q\bar{q}$). The $\pi_1(1400) \to \pi\eta$ can also mix with other exotic mesons with the same quantum numbers but the next known exotic meson is the $\pi_1(1600)$, which is relatively far up on the mass scale. Given this, one could argue that the $\pi_1(1400)$ could be classified with the 14 strange particles with $J^P = 1^-$, thus forming a multiplet 17. However, this is tenuous at best, since the isodoublet strange states with $I(J^P) = 1/2(1^-)$ (K^* and $\bar{K}^*$) could mix with other known K^*'s in the vicinity, e.g. with the $K^*(1410)$ or the $K^*(1680)$. On the other side of the ledger, it is quite possible that a set of 13 states (δ, $\bar{\delta}$, π_1, ω^- and $\bar{\omega}^+$) remain relatively unmixed, because the other exotic states of same flavor may lie far-off in mass.

4 Conclusions

The charge-conjugation properties of $D(p,p)$ representations are well known. Consider an octet $\mathbf{8}$ which is a $D(1,1)$ representation. C operation takes one member of an octet, e.g. K^+, to another one, i.e. K^-, in the same representation. It is clear that the same applies, for example, to the multiplet 27 which is a $D(2,2)$ representation. It is because of this that there exist always a unique C in a given representation. Thus, such $SU(3)$ families are properly labeled by J^{PC}, even though the strange members are not eigenstates of C.

The situation changes drastically when one deals with the representations $D(p,0)$ and $D(0,p)$. (Note that the multiplets $\mathbf{10}$ and $\overline{\mathbf{10}}$ refer to the $D(3,0)$ and $D(0,3)$ representations.) The two multiplets form charge-conjugate pairs, but they belong to different $SU(3)$ representations. A C operation takes one member of $D(p,0)$ into another one in $D(0,p)$ and vice versa. The non-strange members in $D(p,0)$ and $D(0,p)$ combine to form a set of a given J^{PC}. It is, therefore, no longer relevant to speak of an $SU(3)$ family of a given J^{PC}, when one deals with the representations $D(p,0)$ and $D(0,p)$. The representations $D(p,0)$ and $D(0,p)$ together form a multiplet N_p given by

$$N_p = (1+p)\,(2+p) \tag{20}$$

which is derived from (1).

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References

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[3] Our definition is somewhat at variance from the usual convention used for, e.g. u and d quarks. The reader may wish to consult S. U. Chung, ' C- and G-parity: A New Definition and Applications (Version III),' on the web site http://130.199.22.118/reviews.html. For a non-trivial application, the reader may wish to consult S.U. Chung, 'Analysis of $K\bar{K}\pi$ systems,' at the same web site.

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