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Central Production at COMPASS and ALICE —Version IIIb—

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abstract

We work out the kinematics and the production amplitude for central production of resonances in a 2- to 3-body process. The aim is to exhibit the explicit formula for the spin-density matrix for production of the central system.

WORK IN PROGRESS

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1 Introduction

Consider a reaction in which a system 3 is produced centrally, i.e.

$$a + b \to 1 + 3 + 2 \tag{1}$$

For COMPASS, a and 1 are pions, while b and 2 are protons. At the LHC, the initial particles a and b, as well as the final particles 1 and 2, are all protons. The derivation given in this note is general; it applies both to COMPASS and to ALICE at the LHC. Clearly, it applies equally well to other experiments at the LHC, but for a study of the light-quark hadrons with mass less than 3.0 GeV, ALICE is a suitable detector for the spectroscopy, provided the Roman pots were available in the future.

The reader may consult references[1], [2] and [3], for the background material for treating central productions, or more generally the formalism concerning 2- to 3-body reactions.

The produced central system is designated by 3 and its mass by m_3, as shown in Fig. 1: Referring to Fig. 1, we obtain, in the standard 4-momentum notation,

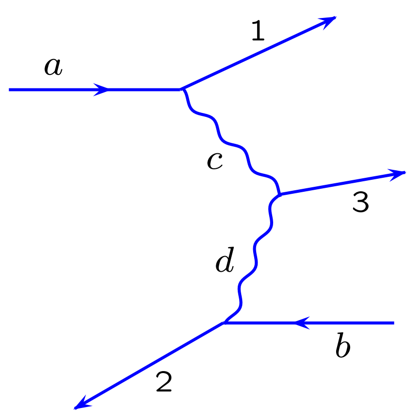

Figure 1: Production of a system 3 from the reaction $a + b \to 1 + 3 + 2$.
Here c and d stand for the exchanged Reggeons (or Pomerons).

$$p_g \equiv p_a + p_b = p_1 + p_3 + p_2 \tag{2a}$$

$$p_c = p_a - p_1 \tag{2b}$$

$$p_d = p_b - p_2 \tag{2c}$$

$$p_3 = p_c + p_d \tag{2d}$$

and

$$\begin{cases} s = (p_a + p_b)^2 = (p_1 + p_2 + p_3)^2 \\ s_{13} = (p_1 + p_3)^2 \quad \text{and} \quad s_{23} = (p_2 + p_3)^2 \\ t_c = p_c^2 = (p_a - p_1)^2 \quad \text{and} \quad t_d = p_d^2 = (p_b - p_2)^2 \\ w^2 = m_3^2 = (p_c + p_d)^2 \end{cases} \tag{3}$$

Here c and d are space-like, i.e. $t_c = p_c^2 \leq 0$ and $t_d = p_d^2 \leq 0$. We assume that the 'masses' $\sqrt{s}$, m_a, m_b, m_1, $m_3 = w$ and m_2 are all fixed and different in general for the problem under consideration.

Let $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_3$ and $\boldsymbol{\kappa}_2$ be the two-dimensional vectors in a plane perpendicular to $\boldsymbol{p}_a$ (defined to be the positive z direction) and/or $\boldsymbol{p}_b$ in the overall CM system. And let q be the one-dimensional momentum along $\boldsymbol{p}_a$ in the overall CM system, and so $-q$ is then directed along the negative $\boldsymbol{p}_a$ or, equivalently, along $\boldsymbol{p}_b$, again in the overall CM system. So, by definition, we have

$$\kappa_i = |\boldsymbol{\kappa}_i|, \quad p_i = |\boldsymbol{p}_i| = \sqrt{\kappa_i^2 + q_i^2} \quad \text{and} \quad \varepsilon_i = \sqrt{p_i^2 + m_i^2} \tag{4}$$

where $i = (a, b, 1, 3, 2)$ for the particles in a 2- to 3-body process, see Fig. 1. Note that p_i now stands for *both* the 4-momentum and the magnitude of the 3-momentum, while ε_i stands for the energy of the particle i in the overall CM system. We define

$$p_i = (\varepsilon_i; \boldsymbol{\kappa}_i, q_i), \qquad \kappa_i \ll q_i \quad \text{where} \quad i = (a, b, 1, 2) \tag{5}$$

where the x and y axes span the the plane perpendicular to $\boldsymbol{p}_a$ and/or $\boldsymbol{p}_b$ and the z axis lies along $\boldsymbol{p}_a$. We can further define[a] the 'transverse mass' w_i, setting p to be the magnitude of

[a] For COMPASS on pp interactions at 190 GeV/c, we have $\sqrt{s} \simeq 18.9$ GeV and $p \simeq 9.40$ GeV. The difference $\varepsilon_a - p = \varepsilon_b - p$ is less than 0.05 GeV. At COMPASS, we have, very roughly, $q_1 \sim 8.0$, $q_2 \sim 9.0$ and $q_3 \sim 1.0$ GeV, respectively, such that $q_2 \sim q_1 + g_3$.

the momentum $\boldsymbol{p}_a$ or $\boldsymbol{p}_b$ in the overall CM system, as follows:

$$\begin{aligned}
w_a &= m_a, \quad w_b = m_b, \quad w_i = \sqrt{m_i^2 + \kappa_i^2}, \quad i = 1, 2, 3 \\
\varepsilon_a &= \sqrt{p^2 + m_a^2} \simeq p + \frac{m_a^2}{2p}, \quad m_a \ll p \\
\varepsilon_b &= \sqrt{p^2 + m_b^2} \simeq p + \frac{m_b^2}{2p}, \quad m_b \ll p \\
\varepsilon_i &= \sqrt{\kappa_i^2 + q_i^2 + m_i^2} = \sqrt{w_i^2 + q_i^2}, \quad i = 1, 2, 3 \\
(w_i \ll q_i) &\to \simeq q_i + \frac{w_i^2}{2q_i}, \quad i = 1,\ 2 \ \text{only}
\end{aligned} \tag{6}$$

In particular, we *do not* assume that $w_3 \ll q_3$. We rewrite all of the momenta once again here,

$$\begin{aligned}
p_a &= (\varepsilon_a; \vec{0}, p) \quad \text{and} \quad p_b = (\varepsilon_b; \vec{0}, -p) \\
p_1 &= (\varepsilon_1; \boldsymbol{\kappa}_1, q_1) \\
p_2 &= (\varepsilon_2; \boldsymbol{\kappa}_2, -q_2) \\
p_3 &= (\varepsilon_3; \boldsymbol{\kappa}_3, q_3)
\end{aligned} \tag{7}$$

where we have set $\boldsymbol{p}_a$ and $\boldsymbol{p}_1$ to be parallel in the limit $\kappa_1 = 0$, while $\boldsymbol{p}_b$ and $\boldsymbol{p}_2$ are also parallel, again in the limit $\kappa_2 = 0$; note that the z components of the latter are both negative, and

$$\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3 = 0 \quad \text{and} \quad q_2 = q_1 + q_3 \tag{8}$$

Specifically, we require that, for some fixed value q_0,

$$\begin{aligned}
q_0 > 0, \quad q_1 > 0 \quad \text{and} \quad q_2 > 0 \\
q_0 \ll q_1, \quad q_0 \ll q_2 \quad \text{and} \quad q_1 \simeq q_2 \\
-q_0 < q_3 < q_0 \quad \text{and so} \quad |q_3| \ll q_1 \text{ or } q_2
\end{aligned} \tag{9}$$

Here q_0 is clearly arbitrary.[b]

We see that

$$\begin{aligned}
\sqrt{s} &= \varepsilon_1 + \varepsilon_3 + \varepsilon_2 \\
&\simeq q_1 + \varepsilon_3 + q_2 = 2q_1 + \varepsilon_3 + q_3 \simeq 2q_2 + \varepsilon_3 + q_3 \gtrsim 2q_1 \simeq 2q_2
\end{aligned} \tag{10}$$

[b] For COMPASS on pp interactions at 190 GeV/c, we may set $q_0 = 1.5$ GeV, approximately 16% of $p \simeq 9.40$ GeV.

assuming that $\varepsilon_3 \ll q_1$ and $\varepsilon_3 \ll q_2$, while

$$\sqrt{s} = \varepsilon_a + \varepsilon_b \simeq 2p, \quad m_a \ll p \quad \text{and} \quad m_b \ll p \tag{11}$$

There is a hierarchy of momenta

$$|q_3| \leq q_0 \ll q_1 \simeq q_2 \lesssim p \simeq \frac{1}{2}\sqrt{s} \tag{12}$$

We are now ready to calculate, for $s \to \infty$, $p \to \infty$, $q_1 \to \infty$ and $q_2 \to \infty$ (but $q_3 = q_2 - q_1$ must remain finite.)

$$\begin{cases} t_c = (p_a - p_1)^2 = (\varepsilon_a - \varepsilon_1)^2 - (p - q_1)^2 - \kappa_1^2 \simeq -\kappa_1^2 \\ t_d = (p_b - p_2)^2 = (\varepsilon_b - \varepsilon_2)^2 - (p - q_2)^2 - \kappa_2^2 \simeq -\kappa_2^2 \end{cases} \tag{13}$$

and, with $m_i \approx \kappa_i \ll q_i$, $(i = 1, 2)$,

$$s_{13} = (p_1 + p_3)^2 = m_1^2 + m_3^2 + 2(\varepsilon_1\varepsilon_3 - q_1 q_3 - \boldsymbol{\kappa}_1 \cdot \boldsymbol{\kappa}_2) \tag{14a}$$

$$\simeq 2q_1(\varepsilon_3 - q_3) + m_1^2 + m_3^2 - 2\boldsymbol{\kappa}_1 \cdot \boldsymbol{\kappa}_2 \tag{14b}$$

$$s_{23} = (p_2 + p_3)^2 = m_2^2 + m_3^2 + 2(\varepsilon_2\varepsilon_3 + q_2 q_3 - \boldsymbol{\kappa}_2 \cdot \boldsymbol{\kappa}_3) \tag{14c}$$

$$\simeq 2q_2(\varepsilon_3 + q_3) + m_2^2 + m_3^2 - 2\boldsymbol{\kappa}_2 \cdot \boldsymbol{\kappa}_3 \tag{14d}$$

where we have used the approximations $\varepsilon_1 \simeq q_1$ and $\varepsilon_2 \simeq q_2$ to arrive at (14b) and (14d). Taking the product of the sub-energies in (14a) and (14b), we see that

$$\begin{aligned} s_{13}s_{23} \simeq{} & 4q_1 q_2 w_3^2 + 2q_1(\varepsilon_3 - q_3)(m_2^2 + m_3^2 - 2\boldsymbol{\kappa}_2 \cdot \boldsymbol{\kappa}_3) \\ & + 2q_2(\varepsilon_3 + q_3)(m_1^2 + m_3^2 - 2\boldsymbol{\kappa}_1 \cdot \boldsymbol{\kappa}_2) \\ & + (m_1^2 + m_3^2 - 2\boldsymbol{\kappa}_1 \cdot \boldsymbol{\kappa}_2)(m_2^2 + m_3^2 - 2\boldsymbol{\kappa}_2 \cdot \boldsymbol{\kappa}_3) \end{aligned} \tag{15a}$$

$$\begin{aligned} s_{13}s_{23} \simeq{} & 4q_1 q_2 w_3^2 + 2q_1(\varepsilon_3 - q_3)(m_2^2 + m_3^2 - 2\boldsymbol{\kappa}_2 \cdot \boldsymbol{\kappa}_3) \\ & + 2q_2(\varepsilon_3 + q_3)(m_1^2 + m_3^2 - 2\boldsymbol{\kappa}_1 \cdot \boldsymbol{\kappa}_2) \end{aligned} \tag{15b}$$

Observe that the first term is dominant; it is a product of q_1 and q_2 and is independent of q_3. In order to derive (15b), we have assumed that, from (14b) and (14d),

$$\begin{aligned} m_1^2 + m_3^2 - 2\boldsymbol{\kappa}_1 \cdot \boldsymbol{\kappa}_2 &\ll 2q_1(\varepsilon_3 - q_3) \\ m_2^2 + m_3^2 - 2\boldsymbol{\kappa}_2 \cdot \boldsymbol{\kappa}_3 &\ll 2q_2(\varepsilon_3 + q_3) \end{aligned} \tag{16}$$

Assuming that these conditions are satisfied, we are able to drop, from ($15a$), the term devoid of q_1 or q_2 to obtain ($15b$).[c] We can rewrite the conditions (16) with the conditions spelled out in (10)

$$\begin{aligned} \varepsilon_3 \ll q_2 \quad &\rightarrow \quad 2q_1(\varepsilon_3 - q_3) \ll 2q_1(q_2 - q_3) \\ \varepsilon_3 \ll q_1 \quad &\rightarrow \quad 2q_2(\varepsilon_3 + q_3) \ll 2q_2(q_1 + q_3) \end{aligned} \tag{17}$$

Combining (16) and (17), we obtain

$$\begin{aligned} m_1^2 + m_3^2 - 2\boldsymbol{\kappa}_1 \cdot \boldsymbol{\kappa}_2 \ll 2q_1(\varepsilon_3 - q_3) \ll 2q_1(q_2 - q_3) \\ m_2^2 + m_3^2 - 2\boldsymbol{\kappa}_2 \cdot \boldsymbol{\kappa}_3 \ll 2q_2(\varepsilon_3 + q_3) \ll 2q_2(q_1 + q_3) \end{aligned} \tag{18}$$

which show that ε_3 and $|q_3|$ cannot be too large or small.

Retaining only the first term in (15) and using (12), we obtain finally,

$$\text{Regge Domain:} \qquad \boxed{s_{13}s_{23} \simeq sw_3^2 = s(m_3^2 + \kappa_3^2)} \tag{19}$$

a well-known relationship.[d] For a given $\sqrt{s}$ and a fixed w_3, we see that $\sqrt{s_{13}}$ and $\sqrt{s_{23}}$ are dependent on each other by a parabola. This is a crucial formula for central production; *it can be used to define the central production*; no other cuts are needed, not even the t cuts. Kaidalov[3] defines the *Regge domain* through the relationship (19). We recapitulate the conditions under which the formulas (13), (15) and (19) are valid:

(1) $m_1 \approx \kappa_1 \approx w_1 \ll p \approx q_1 \simeq \frac{1}{2}\sqrt{s}$
and $-t_c \simeq \kappa_1^2 \approx m_1^2$ remains finite

(2) $m_2 \approx \kappa_2 \approx w_2 \ll p \approx q_2 \approx q_1 \simeq \frac{1}{2}\sqrt{s}$
and $-t_d \simeq \kappa_2^2 \approx m_2^2$ remains finite

(3) $m_3 \approx \kappa_3 \approx w_3$ and $\varepsilon_3^2 = w_3^2 + q_3^2$ where $-q_0 < q_3 < q_0$;
(a) $q_2 = q_1 + q_3$; (b) $q_0 \ll q_1 \simeq q_2 \lesssim p \simeq \frac{1}{2}\sqrt{s}$; (c) $\varepsilon_3 \ll q_1 \simeq q_2$;
(d) $m_1^2 + m_3^2 - 2\boldsymbol{\kappa}_1 \cdot \boldsymbol{\kappa}_2 \ll 2q_1(\varepsilon_3 - q_3) \ll 2q_1(q_2 - q_3)$;

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[c] For a sample selected for central production on pp interactions at 190 GeV/c at COMPASS, this formula is satisfied at about 3% level.

[d] For COMPASS on pp interactions at 190 GeV/c, we have $\sqrt{s} \simeq 18.9$ GeV and $w_3 < 2.1$ GeV. The sample selected for central production satisfies this formula at ~ 6 % level.

(e) $m_2^2 + m_3^2 - 2\boldsymbol{\kappa}_2 \cdot \boldsymbol{\kappa}_3 \ll 2q_2(\varepsilon_3 + q_3) \ll 2q_2(q_1 + q_3)$

It is the third condition, with the subsidiary conditions (b), (c), (d) and (e), which are necessary for the equation (19) to be valid. *We have thus defined the precise conditions for the 'central production of a resonance.'*

2 Kinematics for Central Production

We now go into the rest frame of 3 (3RF), see Fig. 2. We first note that, denoting the 3-momenta by boldface in this frame, the equation (2a) becomes

$$E_a + E_b = E_1 + m_3 + E_2 \tag{20a}$$

$$\boldsymbol{p}_a + \boldsymbol{p}_b = \boldsymbol{p}_1 + \boldsymbol{p}_2 \tag{20b}$$

$$\boldsymbol{p}_c = \boldsymbol{p}_a - \boldsymbol{p}_1 \tag{20c}$$

$$\boldsymbol{p}_d = \boldsymbol{p}_b - \boldsymbol{p}_2 \tag{20d}$$

$$E_c + E_d = m_3 \tag{20e}$$

$$\boldsymbol{p}_c + \boldsymbol{p}_d = 0 \tag{20f}$$

since $\boldsymbol{p}_3 = 0$.

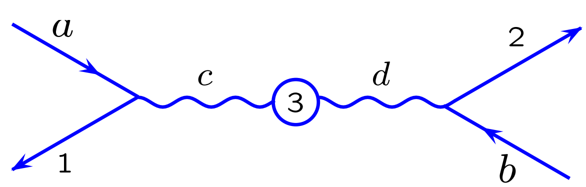

Figure 2: Production of a system 3 via $c + d \to 3$ in the 3 rest frame, where c and d stand for the Reggeons (or Pomerons).

We set up two planes $(1c|a)$ and $(2d|b)$ in the 3RF by specifying their normals, i.e.

$$\boldsymbol{n}_{1a} = \boldsymbol{p}_a \times \boldsymbol{p}_1 \quad \text{and} \quad \boldsymbol{n}_{2b} = \boldsymbol{p}_b \times \boldsymbol{p}_2 \tag{21}$$

We shall fix the plane $(1c|a)$ by specifying $\boldsymbol{p}_c$ to be along the positive z axis and $\boldsymbol{n}_{1a}$ to be along the y axis, i.e. the plane $(1c|a)$ lies in the (xz) plane. The vector $\boldsymbol{p}_1$ can now be specified by the polar angles $\Omega_1 = (\theta_1, 0)$. The vector $\boldsymbol{p}_a$ is given by $\boldsymbol{p}_a = \boldsymbol{p}_c + \boldsymbol{p}_1$. Next the second plane in the problem, the plane $(2d|b)$, can be specified by a rotation around the z axis by an angle ϕ from the plane $(1c|a)$, such that the vectors $\boldsymbol{n}_{1a}$ and $\boldsymbol{n}_{2b}$ intersect with an angle ϕ between them. We note that the two planes now intersect along the z axis by construction. Note that the vector $\boldsymbol{p}_d = -\boldsymbol{p}_c$ lies along the negative z axis. The direction of $\boldsymbol{p}_2$, which lies in the $(2d|b)$, can likewise be fixed by introducting the polar angles $\Omega_2 = (\theta_2, \phi)$. $\boldsymbol{p}_b$ is now given through $\boldsymbol{p}_b = \boldsymbol{p}_d + \boldsymbol{p}_2$. In summary, we can set, in the 3RF,

$$\begin{cases} \boldsymbol{p}_c = (0, 0, p_c), \quad p_c = p_i \\ \boldsymbol{p}_d = (0, 0, p_d), \quad p_d = -p_i \\ \boldsymbol{p}_1 = p_1(\sin\theta_1, 0, \cos\theta_1) \\ \boldsymbol{p}_a = (p_1 \sin\theta_1, 0, p_1\cos\theta_1 + p_i) \\ \boldsymbol{p}_2 = p_2(\sin\theta_2\cos\phi, \sin\theta_2\sin\phi, \cos\theta_2) \\ \boldsymbol{p}_b = (p_2\sin\theta_2\cos\phi, p_2\sin\theta_2\sin\phi, p_2\cos\theta_2 - p_i) \end{cases} \tag{22}$$

It can be shown that the eqations (20b), (20c) and (20d) are well satisfied by the construction above, i.e. the expressions (22) above satisfy (20). There are six parameters, p_i, p_1, θ_1, p_2, θ_2 and ϕ, in the problem. Excluding p_i which is given by $\sqrt{s}$, there are just five parameters in the 3-body phase-space, as expected. This completes the full specification of the reaction (1) in the 3RF.

The invariant phase-space formula takes on the form, in the 3RF,

$$\mathrm{d}\Phi_3 = \left(\frac{\mathrm{d}^3\boldsymbol{p}_1}{2E_1}\right)\left(\frac{\mathrm{d}^3\boldsymbol{p}_2}{2E_2}\right) \tag{23}$$

dropping the multiples of (2π) factors. The differential phase-space element for the particle 3 does not appear since $\boldsymbol{p}_3 = 0$. Using (22), we see that

$$\mathrm{d}\Phi_3 = \left(\frac{1}{2E_1}\right) p_1^2 \mathrm{d}p_1\, \mathrm{dcos}\,\theta_1 \left(\frac{1}{2E_2}\right) p_2^2 \mathrm{d}p_2\, \mathrm{dcos}\,\theta_2\, \mathrm{d}\phi \tag{24}$$

and

$$E_1 = \sqrt{p_1^2 + m_1^2} \quad \text{and} \quad E_2 = \sqrt{p_2^2 + m_2^2} \tag{25}$$

where p_1 and p_2 are once again evaluated in the 3RF.

It is not usual that one evaluates a phase-space element in the rest frame of a final-state particle; we have taken advantage of the fact that the equations (23) and (24) are both Lorentz-invariant expressions, and our development anticipates the partial-wave analyses to be done on the decay of the particle 3, which must be carried out in its rest frame, i.e. in the 3RF.

3 Regge Phenomenology

Here we write down the production amplitudes of Reaction (1), but following the standard prescription for introducing the Reggeized amplitudes. We introduce below two Regge trajectories denoted by the subscripts i and k, but they may very well be the same, i.e. that of the Pomeron

$$\alpha_i(t) = \alpha_k(t) = \alpha(t) = \alpha_0 + \alpha' \, t \tag{26}$$

where $\alpha_0 = 0.081$–1.112 and $\alpha' = 0.25$ GeV^{-2}.

The production amplitudes may be written[3], denoting by $|jm\rangle$ the spin state of the particle 3 in the 3RF,

$$\begin{aligned} {}^{jm}T^{\lambda_1\,\lambda_2}_{\lambda_a\,\lambda_b}(ab \to 1+3+2) = \sum_{i\,k} G_{\lambda_a\,\lambda_1}(t_c) \left(\frac{s_{13}}{s_0}\right)^{\alpha_i(t_c)} \xi[\alpha_i(t_c)] \ \times\ {}^{jm}G_{ik}(t_c, t_d, \phi) \\ \times\, G_{\lambda_b\,\lambda_2}(t_d) \left(\frac{s_{23}}{s_0}\right)^{\alpha_k(t_c)} \xi[\alpha_k(t_d)] \end{aligned} \tag{27}$$

where $s_0 = 1.0$ GeV^2 a standard constant of Regge phenomenology. Once again, the sum on $\{i, k\}$ refers to the possible Regge exchanges at the upper and lower vertices, respectively. The vertex functions $G_{\lambda_a\,\lambda_1}(t_c)$ and $G_{\lambda_b\,\lambda_2}(t_d)$ refer to the decay vertices $(1c|a)$ and $(2d|b)$, while ${}^{jm}G_{ik}(t_c, t_d, \phi)$ refers to the time-reversed decay of $(cd|3)$, see Figs. 1 and 2. The functions $\xi[\alpha(t)]$ are the signature factors, given by

$$\xi[\alpha(t)] = -\frac{1 + \sigma\, e^{-i\,\pi\,\alpha(t)}}{\sin \pi\alpha(t)} \tag{28}$$

where σ is the signature; if i and k both refer to the Pomeron, then we have $\sigma_i = \sigma_k = \sigma = +1$.

4 Decay Amplitudes

Our task here is to work out the decay amplitudes in the 3RF, i.e. $a \to 1 + c$, $b \to 2 + d$ and $3 \to c + d$. For the topics covered here, the reader may consult the references [4] and [5].

For the plane $(1c|a)$, we go to the aRF from the 3RF and take the canonical axes, i.e. we take the same coordinate system defined in the previous section. here we consider the tow-body decay $a \to 1 + c$. The decay amplitude is, denoting the decay-coupling amplitudes by F,

$$A(a \to 1 + c) \sim \, {}^{j_a}F^{j_1\, j_c}_{\lambda_1\, \lambda_c}\, d^{j_a}_{\lambda_a,\, \lambda_1 - \lambda_c}(\theta'_1) \tag{29}$$

where $\Omega'_1 = (\theta'_1, 0)$ and θ'_1 is the angle between $\vec{a}$ in the $(a{+}b)$RF and the vector p'_1 in the aRF, reached via pure time-like Lorentz transformation from the $(a{+}b)$RF. By the identical procedure, we can write down the decay amplitude for $b \to 2 + d$, again in the bRF and defined in the plane $(2d|b)$, denoting the decay-coupling amplitudes again by F,

$$A(b \to 2 + d) \sim \, {}^{j_b}F^{j_2\, j_d}_{\lambda_2\, \lambda_d}\, D^{j_b\, *}_{\lambda_b,\, \lambda_2 - \lambda_d}(\Omega'_2) \tag{30}$$

where the angles $\Omega'_2 = (\theta'_2, \phi)$ is defined such that ϕ ia still the same angle defined in the previous section but θ'_2 is the angle between $\vec{b}$ in the $(a{+}b)$RF and the vector p'_2 in the bRF. We note that the spin components λ_a and λ_b are the helicities along $\vec{p}_a$ and $\vec{p}_b$ in the $(a{+}b)$RF.

We now work on the problem of writng down the amplitude for the process $c + d \to 3$. For the purpose we deal with the time-reversed process $3 \to c + d$. The relevant vectors are already defined (22). We are ready to work out the decay process $3 \to c + d$ in the 3RF:

$$A(3 \to c + d) \sim \, {}^{j}F^{j_c\, j_d}_{\lambda_c\, \lambda_d}\, D^{j\, *}_{m,\, \lambda_c - \lambda_d}(0, 0, 0) \; = \; {}^{j}F^{j_c\, j_d}_{\lambda_c\, \lambda_d} \tag{31}$$

where F is once again the decay-coupling amplitude, and $\Omega_0 = (0, 0)$ specifies the orientation of the vector $\vec{p}_c$ or $-\vec{p}_d$ in the 3RF just defined. The D-function is zero unless $m = \lambda_c - \lambda_d$, where m is the spin component of 3 in this frame. Here we emphasize that the "body-fixed" helicites λ_c and λ_d are those defined (29) and (30); it is for this reason that the same notations are used for both.

The overall amplitude is the product of the three amplitudes defined above

$$A^j_m(\Omega'_1, \Omega'_2) = \sum_{j_c\, j_d} \sum_{\lambda_c \lambda_d} A(a \to 1 + c)\, A(b \to 2 + d)\, A^\dagger(3 \to c + d) \tag{32}$$

where we have introduced a Hermitian conjugate of $A(3 \to c + d)$ to represent the time-reversed process $c + d \to 3$. Note that the particles a, b, 1 and 2 are external particles; their helicities will be summed over at the cross-section level after taking the absolute square of the amplitudes. We should emphasize that the three sets of angles Ω_0, Ω_1' and Ω_2' are all defined three Lorentz frames as specified previously. We note in addition that there is a summation over the spins j_c and j_d. However, they will be transformed beyond recognition by the process of Reggeization of the exchanged particles. There still remains the possible choice of the spin sets. For example, if we suppose that the Pomerons are responsible for both, then the possible spin-parity sets are $J^{PC} = 2^{++}$, 4^{++} or 6^{++}, or 2^{++} only for simplicity to represent the Pomeron state.

The helicity-coupling amplitudes F satisfy the following relationship from parity conservation in the decay $|jm\rangle \to |s_1\,\lambda_1\rangle + |s_2\,\lambda_2\rangle$

$$ {}^{j}F^{s_1\,s_2}_{-\lambda_1\,-\lambda_2} = \eta_j\,\eta_1\eta_2\,(-)^{j-s_1-s_2}\;{}^{j}F^{s_1\,s_2}_{\lambda_1\,\lambda_2} = \nu_j\,\bar{\nu}_1\bar{\nu}_2\;{}^{j}F^{s_1\,s_2}_{\lambda_1\,\lambda_2} \tag{33} $$

where η_k is the intrinsic parity with $k = \{\,j, 1, 2\,\}$, and ν_k and $\bar{\nu}_k$ are the 'naturality' of the particle k,

$$ \nu_k = \eta_k(-)^{j_k}, \qquad \bar{\nu}_k = \eta_k(-)^{-j_k} \qquad \text{for} \quad \text{bosons and fermions} \tag{34} $$

where ν_k and $\bar{\nu}_k$ are real for bosons and imaginary for fermions. The two-body decay processes, $j \to 1 + 2$, always involve three particles, in which zero or two can be fermions but all three can never be fermions. As a result, a product of two ν's must be real such that

$$ \nu_j\,\bar{\nu}_1 = \pm 1 \quad \text{or} \quad \nu_j\,\bar{\nu}_2 = \pm 1 \quad \text{or} \quad \bar{\nu}_1\,\bar{\nu}_2 = \pm 1 $$

This covers all the cases of fermions in the problem, and as a result we see that the factor in (33) must obey

$$ \nu_j\,\bar{\nu}_1\bar{\nu}_2 = \pm 1 $$

which is as expected.

We note that the coupling constants F's can made real under certain general conditions (time-reversal invariance), see Section 5.2 of Reference[5].

Finally, we give here the angles θ_1' and θ_2' in terms of the Lorentz-invariant variables in the problem. We start with

$$
a\text{RF} \to \begin{cases} \quad s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2m_a\, E_b' \quad \to \quad p_b' = \sqrt{(E_b')^2 - m_b^2} \\ \quad t_c = (p_a - p_1)^2 = m_a^2 + m_1^2 - 2m_a\, E_1' \quad \to \quad p_1' = \sqrt{(E_1')^2 - m_1^2} \\ E_g' = m_a + E_b' \\ \boldsymbol{p}_g' = \boldsymbol{p}_b' = \boldsymbol{p}_1' + \boldsymbol{p}_3' + \boldsymbol{p}_2' \end{cases} \tag{35}
$$

We see that, from (2),

$$
\begin{aligned} s_{23} &= (p_2 + p_3)^2 = (p_g - p_1)^2 \\ a\text{RF} \to &= s + m_1^2 - 2(E_g'\, E_1' - p_g'\, p_1' \cos\theta_1') \\ &= s + m_1^2 - 2\Big[(m_a + E_b')\, E_1' - p_b'\, p_1' \cos\theta_1'\Big] \end{aligned} \tag{36}
$$

In summary, the angle θ_1' depends on three variables s, t_c and s_{23}. Likewise, we note that

$$
b\text{RF} \to \begin{cases} \quad s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2m_b\, E_a' \quad \to \quad p_a' = \sqrt{(E_a')^2 - m_a^2} \\ \quad t_d = (p_b - p_2)^2 = m_b^2 + m_2^2 - 2m_b\, E_2' \quad \to \quad p_2' = \sqrt{(E_2')^2 - m_2^2} \\ E_g' = m_b + E_a' \\ \boldsymbol{p}_g' = \boldsymbol{p}_a' = \boldsymbol{p}_1' + \boldsymbol{p}_3' + \boldsymbol{p}_2' \end{cases} \tag{37}
$$

We see that, again from (2),

$$
\begin{aligned} s_{13} &= (p_1 + p_3)^2 = (p_g - p_2)^2 \\ b\text{RF} \to &= s + m_2^2 - 2(E_g'\, E_2' - p_g'\, p_2' \cos\theta_2') \\ &= s + m_1^2 - 2\Big[(m_b + E_a')\, E_2' - p_a'\, p_2' \cos\theta_2'\Big] \end{aligned} \tag{38}
$$

So the angle θ_2' can be evaluated from s, t_d and s_{13}.

We remark that the angles θ_1' and θ_2' are determined through their cosines, so the angles can be evaluated in the range $\{0 \to \pi\}$.

5 Reggeized Production Amplitudes

We now combine the results of Sections 3 and 4 and write down the Reggeized production amplitudes. For the purpose we modify (32) to conform to the double-Reggeon exchange amplitude (27)

$$A^j_m(\Omega'_1,\Omega'_2) = \sum_{i\,k}\sum_{j_c\,j_d}\sum_{\lambda_c\lambda_d} A(a\to 1+c)\, B_i(t_c)\, \left(\frac{s_{13}}{s_0}\right)^{\alpha_i(t_c)} \xi[\alpha_i(t_c)]$$
$$\times\; A^\dagger(3\to c+d) \tag{39}$$
$$\times \left(\frac{s_{23}}{s_0}\right)^{\alpha_k(t_d)} \xi[\alpha_k(t_d)]\, A(b\to 2+d)\, B_k(t_d)$$

Here we have suppressed helicity indices in A for clarity. The three amplitudes $A(a)$, $A(b)$ and $A(3)$ play the role of the vertex functions G defined in (27).

The functions $B_i(t_c)$ and $B_k(t_d)$ are phenomenological t dependence we introduce, setting $t' = |t| - |t|_{\min}$,

$$B_i(t_c) = \left(\frac{t'_c}{t^0_c}\right)^{-|\lambda_c|/2} \quad \text{and} \quad B_k(t_d) = \left(\frac{t'_d}{t^0_d}\right)^{-|\lambda_d|/2} \quad \text{where} \quad \lambda_c - \lambda_d \equiv m \tag{40}$$

where the ket state $|jm\rangle$ stands for the spin state of the particle 3 in its rest frame. Here $t^0_c = t^0_d = +1.0$ GeV^2 (phenomenological). But, according to Boreskov[2], they are quite different,

$$B_i(t_c) = \left(\frac{-t_c}{s_{13}}\right)^{-|\lambda_c|/2} \quad \text{and} \quad B_k(t_d) = \left(\frac{-t_d}{s_{23}}\right)^{-|\lambda_d|/2} \tag{41}$$

Together with the sub-energy dependent factors in (39), following Boreskov, we see that the overall t dependence is, with $\alpha = \alpha_0 + \alpha' t'$,

$$A^j_m(\Omega'_1,\Omega'_2) \sim \sum_{\lambda_c\,\lambda_d} A(a\to 1+c)\,\xi[\alpha(t_c)] \left(\frac{-t_c}{s_{13}}\right)^{-|\lambda_c|/2} \exp\left[-\ln\left(\frac{s_{13}}{s_0}\right)(\alpha_{0\,i} + \alpha'_i\, t'_c)\right]$$
$$\times\; A^\dagger(3\to c+d) \tag{42}$$
$$\times\; A(b\to 2+d)\xi[\alpha(t_d)] \left(\frac{-t_d}{s_{23}}\right)^{-|\lambda_d|/2} \exp\left[-\ln\left(\frac{s_{23}}{s_0}\right)(\alpha_{0\,k} + \alpha'_k\, t'_d)\right]$$

where $\lambda_c - \lambda_d = m$ and assuming $i = k$ and suppressing the summation over i and k for clarity. We now replace the amplitudes $A(a\to 1+c)$, $A^\dagger(3\to c+d)$ and $A(b\to 2+d)$ with the decay amplitudes given in Section 4, and obtain

$$
\begin{aligned}
A^j_m(\Omega'_1,\Omega'_2) \sim \sum_{i\,k}\sum_{\lambda_c\,\lambda_d} & {}^{j_a}F^{j_1\,j_c}_{\lambda_1\,\lambda_c}\, d^{j_a}_{\lambda_a,\,\lambda_1-\lambda_c}(\theta'_1)\,\xi[\alpha(t_c)]\left(\frac{-t_c}{s_{13}}\right)^{-|\lambda_c|/2} \\
& \times \exp\left[-\ln\left(\frac{s_{13}}{s_0}\right)(\alpha_{0\,i}+\alpha'_i\,t'_c)\right]\,{}^{j}F^{j_c\,j_d}_{\lambda_c\,\lambda_d} \\
& \times\ {}^{j_b}F^{j_2\,j_d}_{\lambda_2\,\lambda_d}\,D^{j_b\,*}_{\lambda_b,\,\lambda_2-\lambda_d}(\phi,\theta'_2,0) \\
& \times\ \xi[\alpha(t_d)]\left(\frac{-t_d}{s_{23}}\right)^{-|\lambda_d|/2}\exp\left[-\ln\left(\frac{s_{23}}{s_0}\right)(\alpha_{0\,k}+\alpha'_k\,t'_d)\right] \\
& \qquad\qquad\qquad\qquad \text{where}\quad \lambda_c-\lambda_d=m
\end{aligned}
\tag{43}
$$

Here we note that, if we take the helicity-coupling amplitudes F to be real, then the complex nature of the expression above comes from the factor $\exp[i\,\lambda_b\,\phi]$ contained in the D-function above and the signature factors $\xi[\alpha(t_c)]$ and $\xi[\alpha(t_d)]$. We impose parity conservation in the decay through the following technique

$$
\begin{aligned}
A^j_m(\Omega'_1,\Omega'_2) &= \frac{1}{2}\left[(\text{original})+(\lambda_k\to-\lambda_k\ \text{and}\ m\to-m)\right], \qquad k=\{a,b,1,2,c,d\} \\
&\sim \frac{1}{2}\sum_{i\,k}\sum_{\lambda_c\,\lambda_d} {}^{j_a}F^{j_1\,j_c}_{\lambda_1\,\lambda_c}\, d^{j_a}_{\lambda_a,\,\lambda_1-\lambda_c}(\theta'_1)\,\xi[\alpha(t_c)]\left(\frac{-t_c}{s_{13}}\right)^{-|\lambda_c|/2} \\
&\qquad \times \exp\left[-\ln\left(\frac{s_{13}}{s_0}\right)(\alpha_{0\,i}+\alpha'_i\,t'_c)\right]\,{}^{j}F^{j_c\,j_d}_{\lambda_c\,\lambda_d} \\
&\qquad \times\ {}^{j_b}F^{j_2\,j_d}_{\lambda_2\,\lambda_d}\,D^{j_b\,*}_{\lambda_b,\,\lambda_2-\lambda_d}(\phi,\theta'_2,0) \\
&\qquad \times\ \xi[\alpha(t_d)]\left(\frac{-t_d}{s_{23}}\right)^{-|\lambda_d|/2}\exp\left[-\ln\left(\frac{s_{23}}{s_0}\right)(\alpha_{0\,k}+\alpha'_k\,t'_d)\right] \\
&+ \frac{1}{2}\sum_{i\,k}\sum_{\lambda_c\,\lambda_d} {}^{j_a}F^{j_1\,j_c}_{-\lambda_1\,-\lambda_c}\, d^{j_a}_{-\lambda_a,\,-\lambda_1+\lambda_c}(\theta'_1)\,\xi[\alpha(t_c)]\left(\frac{-t_c}{s_{13}}\right)^{-|\lambda_c|/2} \\
&\qquad \times \exp\left[-\ln\left(\frac{s_{13}}{s_0}\right)(\alpha_{0\,i}+\alpha'_i\,t'_c)\right]\,{}^{j}F^{j_c\,j_d}_{-\lambda_c\,-\lambda_d} \\
&\qquad \times\ {}^{j_b}F^{j_2\,j_d}_{-\lambda_2\,-\lambda_d}\,D^{j_b\,*}_{-\lambda_b,\,-\lambda_2+\lambda_d}(\phi,\theta'_2,0) \\
&\qquad \times\ \xi[\alpha(t_d)]\left(\frac{-t_d}{s_{23}}\right)^{-|\lambda_d|/2}\exp\left[-\ln\left(\frac{s_{23}}{s_0}\right)(\alpha_{0\,k}+\alpha'_k\,t'_d)\right]
\end{aligned}
\tag{44}
$$

where $\lambda_c - \lambda_d = m$. Applying parity conservationin in the decay vertices and the well-known formula for $d^j_{m,\,m'}(\theta)$ under interchange of $m \to -m$ and $m' \to -m'$, and noting that j_c and

j_d must be integers always, we obtain

$$\begin{aligned}
A^j_m(\Omega'_1,\Omega'_2) \sim \sum_{i\,k}\sum_{\lambda_c\,\lambda_d} {}^{j}F^{j_c\,j_d}_{\lambda_c\,\lambda_d} \\
\times \frac{1}{2}\Big\{ \exp(i\,\lambda_b\phi) + \exp(-i\,\lambda_b\phi)\,(\nu_a\,\bar\nu_1)\,(-)^{\lambda_a-\lambda_1}\,(\nu_b\,\bar\nu_2)\,(-)^{\lambda_b-\lambda_2}\,\nu(-)^m \Big\} \\
\times\ {}^{j_a}F^{j_1\,j_c}_{\lambda_1\,\lambda_c}\, d^{j_a}_{\lambda_a,\,\lambda_1-\lambda_c}(\theta'_1)\,\xi[\alpha(t_c)]\left(\frac{-t_c}{s_{13}}\right)^{-|\lambda_c|/2} \\
\times \exp\left[-\ln\left(\frac{s_{13}}{s_0}\right)(\alpha_{0\,i}+\alpha'_i\,t'_c)\right] \\
\times\ {}^{j_b}F^{j_2\,j_d}_{\lambda_2\,\lambda_d}\, d^{j_b}_{\lambda_b,\,\lambda_2-\lambda_d}(\theta'_2)\,\xi[\alpha(t_d)]\left(\frac{-t_d}{s_{23}}\right)^{-|\lambda_d|/2} \\
\times \exp\left[-\ln\left(\frac{s_{23}}{s_0}\right)(\alpha_{0\,k}+\alpha'_k\,t'_d)\right]
\end{aligned} \tag{45}$$

where once again $\lambda_c - \lambda_d = m$ and ν is the naturality of the system 3 in a state $|jm\rangle$. Here we have taken into account the fact that the particle pairs $(a,1)$ and $(b,2)$ can be either both fermions or both bosons but never a mixture, while the three particles $(c,d,3)$ are always neutral bosons with integer spins. We have carefully arranged the six factors which accompany $\exp(-i\,\lambda_b\phi)$ such that each of them is always an integer $=\pm1$ and therefore the product of the six factors is equal to ±1 overall. We need to point out that the factors

$$\nu_a\,\bar\nu_1 = +1 \quad \text{and} \quad \nu_b\,\bar\nu_2 = +1$$

If the identity of the initial and final particles does not change (most of the practical examples for central production falls into this category). The exceptions are, for example, $(a,1)$ corresponds to (p,Δ^+) or (π^-,ρ^-) in which case $\nu_a\,\bar\nu_1 = -1$, and the same applies to $\nu_b\,\bar\nu_2 = -1$. The first factor of (45) shows that the ϕ dependence is given either by $\cos\phi$ or by $i\sin\phi$; the precise functional dependence depends on the helicities and the naturalities as shown in the curly brackets. *This is the main result of this note.*

We remark that, if the production for 3 is dominated by the Pomerons with $J^{PC} = 2^{++}$, then we can set $j_c = j_d = 2$ and furthermore $\lambda_c = \lambda_d = 0$ as a first approximation, i.e. the states with $|jm\rangle$ for which $m = \pm1$ is relatively small compared to those with $m = 0$. Then we obtain

$$
\begin{aligned}
A^j_0(\Omega'_1,\Omega'_2) \sim\ & {}^{j}F^{2\,2}_{0\,0} \\
& \times \frac{1}{2}\Big\{ \exp(i\,\lambda_b\phi) + \exp(-i\,\lambda_b\phi)\,(\nu_a\,\bar\nu_1)\,(-)^{\lambda_a-\lambda_1}\,(\nu_b\,\bar\nu_2)\,(-)^{\lambda_b-\lambda_2}\ \nu \Big\} \\
& \times\ \Bigg\{ {}^{j_a}F^{j_1\,2}_{\lambda_1\,0}\, d^{j_a}_{\lambda_a,\,\lambda_1}(\theta'_1)\,\xi[\alpha(t_c)] \\
& \qquad\quad \times \exp\left[-\ln\left(\frac{s_{13}}{s_0}\right)(\alpha_0+\alpha'\,t'_c)\right] \\
& \qquad\quad \times\ {}^{j_b}F^{j_2\,2}_{\lambda_2\,0}\, d^{j_b}_{\lambda_b,\,\lambda_2}(\theta'_2)\,\xi[\alpha(t_d)] \\
& \qquad\qquad\quad \times \exp\left[-\ln\left(\frac{s_{23}}{s_0}\right)(\alpha_0+\alpha'\,t'_d)\right]\ \Bigg\}
\end{aligned}
\tag{46}
$$

where we have taken out the helicity-coupling constant (and real) ${}^{j}F^{2\,2}_{0\,0}$ out in front, to indicate that this is the *only* j-dependent factor in the formula. In other words, the expression within the second curly bracket is *independent* of j and hence it is merely a proportionality factor. But we emphasize, however, that it does depend on the naturality ν of the central system 3 (note that $\nu=+1$ always if the decay product of 3 consists of two pseudoscalars.)

It is clear that all the physics in the central production is contained within the curly brackets, and the spin-density matrix is simply a bilinear product of unknown real constants ${}^{j}F^{2\,2}_{0\,0}$ for different j's. So the spin density-matrix is real and of rank-1, but the elements are unknown. We note that, with w standing for the effective mass of 3, the cross section is proportional to

$$
\begin{aligned}
\mathrm{d}\sigma &\sim \sum_{\substack{\lambda_a\,\lambda_b\\ \lambda_1\,\lambda_2}} \int \left(\frac{{p'_1}^2\mathrm{d}p'_1\mathrm{d}\cos\theta'_1}{2E'_1}\right)_{a\mathrm{RF}} \left(\frac{{p'_2}^2\mathrm{d}p'_2\mathrm{d}\cos\theta'_2\mathrm{d}\phi}{2E'_2}\right)_{b\mathrm{RF}} \left|A^j_0(\Omega'_1,\Omega'_2)\right|^2 \\
&\sim \sum_{\substack{\lambda_a\,\lambda_b\\ \lambda_1\,\lambda_2}} \int \left(\frac{\mathrm{d}\kappa_1^2\mathrm{d}q'_1}{2E'_1}\right)_{a\mathrm{RF}} \left(\frac{\mathrm{d}\kappa_2^2\mathrm{d}q'_2\mathrm{d}\phi}{2E'_2}\right)_{b\mathrm{RF}} \left|A^j_0(\Omega'_1,\Omega'_2)\right|^2
\end{aligned}
\tag{47}
$$

where the Lorentz invariance of the two-dimensional vectors $\vec\kappa_{1,2}$ along the direction of $\vec p_a+\vec p_b$ has been used above, since they are perpendicular to the direction of the Lorentz transformation. We require the vector $\vec\kappa_1$ to point along the x-axis and the vector $\vec\kappa_2$, which lies in the xy-plane, to have an angle ϕ from the x-axis. Note that $q'_1(q'_2)$ is the longitudinal momentum of $\vec{p}_1(\vec{p}_2)$ evaluated in the aRF(bRF). We recall that $t_c \simeq \kappa_1^2$ and $t_d \simeq \kappa_2^2$, so that

$$
\begin{aligned}
G(w, t_c, t_d, \phi) &\sim \frac{\mathrm{d}\sigma}{\mathrm{d}t_c\,\mathrm{d}t_d\,\mathrm{d}\phi} \\
&\sim \sum_{\substack{\lambda_a\,\lambda_b\\ \lambda_1\,\lambda_2}} \int \left(\frac{\mathrm{d}q_1'}{2E_1'}\right)_{a\mathrm{RF}} \left(\frac{\mathrm{d}q_2'}{2E_2'}\right)_{b\mathrm{RF}} \left|A_0^j(\Omega_1', \Omega_2')\right|^2
\end{aligned}
\tag{48}
$$

We emphasize here that the phase-space factor for the particle 1(2) is evaluated in the $a(b)$RF; thus we have taken advantage of the Lorentz invariance of the phase-space factors—the first factor in the aRF and the second in the bRF—a highly unusual formulation of the problem adapted in this note. The integration above is considerably facilitated by noting that, simply rewriting (46),

$$
\begin{aligned}
A_0^j(\Omega_1', \Omega_2') =\ & {}^{j}F_{0\,0}^{2\,2} \\
&\times \left\{ \xi[\alpha(t_c)] \times \exp\left[-\ln\left(\frac{s_{13}}{s_0}\right)(\alpha_0 + \alpha'\, t_c')\right] \right\} \\
&\times \left\{ \xi[\alpha(t_d)] \times \exp\left[-\ln\left(\frac{s_{23}}{s_0}\right)(\alpha_0 + \alpha'\, t_d')\right] \right\} \\
&\times \frac{1}{2}\Big\{ \exp(i\,\lambda_b\phi) + \exp(-i\,\lambda_b\phi)\,(\nu_a\,\bar{\nu}_1)\,(-)^{\lambda_a-\lambda_1}\,(\nu_b\,\bar{\nu}_2)\,(-)^{\lambda_b-\lambda_2}\;\nu \Big\} \\
&\times \left\{ {}^{j_a}F_{\lambda_1\,0}^{j_1\,2}\; d^{j_a}_{\lambda_a,\,\lambda_1}(\theta_1')\Big|_{a\mathrm{FR}} \times\ {}^{j_b}F_{\lambda_2\,0}^{j_2\,2}\; d^{j_b}_{\lambda_b,\,\lambda_2}(\theta_2')\Big|_{b\mathrm{FR}} \right\}
\end{aligned}
\tag{49}
$$

So the A function has been broken up into four factors, and is shown as curly brackets above. The first factor depends on ϕ only; the second and the third on t_c and t_d, respectively; Only the fourth factor is dependent on q_1' and q_2', the variables of integration indicated in (48).

For completeness, we exhibit (45) again here, the general formula for A in which the helicities for c and d are allowed to take on arbitrary values, but modified to highlight differences in the factors

$$
\begin{aligned}
A_m^j(\Omega_1', \Omega_2') \sim & \sum_{i\,k} \sum_{\lambda_c\,\lambda_d} {}^{j}F_{\lambda_c\,\lambda_d}^{j_c\,j_d} \\
&\times \frac{1}{2}\Big\{ \exp(i\,\lambda_b\phi) + \exp(-i\,\lambda_b\phi)\,(\nu_a\,\bar{\nu}_1)\,(-)^{\lambda_a-\lambda_1}\,(\nu_b\,\bar{\nu}_2)\,(-)^{\lambda_b-\lambda_2}\;\nu(-)^m \Big\} \\
&\times \left\{ {}^{j_a}F_{\lambda_1\,\lambda_c}^{j_1\,j_c} \left(\frac{-t_c}{s_{13}}\right)^{-|\lambda_c|/2} \times \xi[\alpha(t_c)] \times \exp\left[-\ln\left(\frac{s_{13}}{s_0}\right)(\alpha_{0\,i} + \alpha_i'\, t_c')\right] \right\} \\
&\times \left\{ {}^{j_b}F_{\lambda_2\,\lambda_d}^{j_2\,j_d} \left(\frac{-t_d}{s_{23}}\right)^{-|\lambda_d|/2} \times \xi[\alpha(t_d) \times \exp\left[-\ln\left(\frac{s_{23}}{s_0}\right)(\alpha_{0\,k} + \alpha_k'\, t_d')\right] \right\} \\
&\times \left\{ d^{j_a}_{\lambda_a,\,\lambda_1-\lambda_c}(\theta_1')\Big|_{a\mathrm{RF}} \times\ d^{j_b}_{\lambda_b,\,\lambda_2-\lambda_d}(\theta_2')\Big|_{b\mathrm{RF}} \right\}
\end{aligned}
\tag{50}
$$

with the constraint $m = \lambda_c - \lambda_d$. This is a general formula which can be applied to any Regge exchanges for c and d with arbitrary helicities. The formula consists of four curly brackets: the first depends on ϕ only; the second and the third on t_c and t_d, respectively; and the fourth on the the longitudinal momenta for the particles 1 and 2.

The next level of approximation[e] would involve setting $\lambda_c = \pm 1$, $\lambda_d = \pm 1$ $m = \pm 1$ subject to the constraint $m = \lambda_c - \lambda_d$. One obvious way to achieve this would be to set, for $m = 0, \pm 1$,

$$\begin{aligned} \{m, \lambda_c, \lambda_d\} &= \{1,1,0\} \quad \text{and} \quad \{1,0,-1\} \\ &= \{0,0,0\}, \quad \{0,1,1\} \quad \text{and} \quad \{0,-1,-1\} \\ &= \{-1,-1,0\} \quad \text{and} \quad \{-1,0,1\} \end{aligned} \tag{51}$$

6 Spin-Density Matrix for Particle 3

We now let τ to stand the decay of 3 into $n \geq 2$ particles. Then the overall amplitude for the production and decay of a system 3 is

$$\mathcal{A}^j_m(w, \Phi_3, \tau) = A^j_m(\Omega'_1, \Omega'_2)\, A^{j\,*}_m(\tau) \tag{52}$$

where w is the effective mass of the system 3. The phase-space factor Φ_3 is the usual Lorentz-invariant element which depend on the angles Ω'_1 and Ω'_2 as well as appropriate momenta to fully specify the production process. If we make the simplifying assumption that the decay amplitude $A^{j\,*}_m(\tau)$ does not depend on the production variables except the mass w, then we may write

$$\mathcal{A}^j_m(w, \Phi_3, \tau) = A^j_m(\Omega'_1, \Omega'_2)\, A^{j\,*}_m(w, \tau) \tag{53}$$

So we have made an assumption that the amplitude $A^{j\,*}_m(w, \tau)$ does not depend on any of the production variables in $A^j_m(\Omega'_1, \Omega'_2)$. This cannot be true in general but merely a practical simplification.

[e] We know that at $\sim 10\%$ level there are $\lambda = \pm 1$ components present in the COMPASS data.

The differential cross section is, summing over the 'external' helicities,

$$
\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} &\sim \sum_{\substack{\lambda_a,\lambda_b\\ \lambda_1,\lambda_2}} \int \mathrm{d}\Phi_3 \left| \sum_{j\,m} \mathcal{A}^j_m(w,\Phi_3,\tau) \right|^2 \\
&\sim \sum_{\substack{\lambda_a,\lambda_b\\ \lambda_1,\lambda_2}} \sum_{\substack{j\,m\\ j'\,m'}} \int \mathrm{d}\Phi_3 \; A^j_m(\Omega'_1,\Omega'_2)\, A^{j\,*}_m(w,\tau) A^{j'\,*}_{m'}(\Omega'_1,\Omega'_2)\, A^{j'}_{m'}(w,\tau) \qquad (54)\\
&\sim \sum_{\substack{\lambda_a,\lambda_b\\ \lambda_1,\lambda_2}} \sum_{\substack{j\,m\\ j'\,m'}} A^{j\,*}_m(w,\tau)\, A^{j'\,*}_{m'}(w,\tau) \int \mathrm{d}\Phi_3 \; A^j_m(\Omega'_1,\Omega'_2)\, A^{j'\,*}_{m'}(\Omega'_1,\Omega'_2)
\end{aligned}
$$

where the differential element of the phase $\mathrm{d}\Phi_3$ has already been given in (23) and (24). So we have come up with a reasonable model for the production and decay of the system 3. We may define the spin-density matrix in the ususal way

$$
\rho^*_{jm;\,j'm'} = \sum_{\substack{\lambda_a,\lambda_b\\ \lambda_1,\lambda_2}} \int \mathrm{d}\Phi_3 \; A^j_m(\Omega'_1,\Omega'_2)\, A^{j'\,*}_{m'}(\Omega'_1,\Omega'_2) \qquad (55)
$$

From (48) we see that the expression above becomes

$$
\rho^*_{jm;\,j'm'}(t_c,t_d,\phi) \sim \sum_{\substack{\lambda_a,\lambda_b\\ \lambda_1,\lambda_2}} \int \left(\frac{\mathrm{d}q'_1}{2E'_1}\right)_{a\mathrm{RF}} \left(\frac{\mathrm{d}q'_2}{2E'_2}\right)_{b\mathrm{RF}} A^j_m(\Omega'_1,\Omega'_2)\, A^{j'\,*}_{m'}(\Omega'_1,\Omega'_2) \qquad (56)
$$

Note that the decay coupling constants ${}^{j_a}F^{j_1\,j_c}_{\lambda_1\,\lambda_c}$, ${}^{j_b}F^{j_2\,j_d}_{\lambda_2\,\lambda_d}$, ${}^{j}F^{j_c\,j_d}_{\lambda_c\,\lambda_d}$ (for different j's) are unknown, so one can only explore the range of density matrix assuming certain values for the coupling constants. The amplitudes A^j_m and $A^{j'}_{m'}$ above have internal summation on λ_c and λ_d with $m = \lambda_c - \lambda_d$, and on λ'_c and λ'_d with $m' = \lambda'_c - \lambda'_d$.

The formula (34) can be given in a compact expression in terms of the density matrix

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} \sim \sum_{\substack{j\,m\\ j'\,m'}} \rho_{jm;\,j'm'} \; A^{j\,*}_m(w,\tau)\, A^{j'}_{m'}(w,\tau) \qquad (57)
$$

Note that we have here taken a complex conjugate of (34).

The remaining task is to carry out the integral in (56) and find out what terms are important after the phase-space integral, and thus gain insight into the process of 2- to 3-body reactions for a partial-wave anlysis of the central system 3.

7 Conclusions

We have here worked out the sub-energy formula at three levels of approximations:

(a) the formula (15a) at the first-level approximation;
(b) the formula (15b) at the second-level approximation;
(c) the formula (19) at the third-level approximation.

It is the third-level approximation which is widely quoted in the literature, see [2] and [3]. The formula (19) is not very well satisfied at COMPASS energies. Even with (15), the first-level approximation, the equality is violated at $(\simeq 3.0\,\text{GeV})^4$. So, from this point of view, it appears that the central production *cannot* be defined through (15a), (15b) or (19). Or, to put it in other words, the COMPASS energy is simply not high enough for the conditions of central production, conventionally defined and spelled out in Section 1, to be valid.

But for a study of light-quark spectroscopy the formula (19) should be well statisfied for ALICE. If the Roman pots were available for ALICE, a program of light-quark spectroscopy of mesons with mass less than 3.0 GeV can be carried out—with an enormous increase in sensitivity for the light-quark mesons in the range 2.0 GeV–3.0 GeV, which are still poorly known. It is possible, in addition, that the spectroscopy of charmonium states, in the mass range of 3.0 to 4.5 GeV, could be handled as well with the ALICE detector, competitive with other experimental facilities elsewhere in the world.

The spin-density matrix is given in (55) and the general formula for the amplitudes A^j_m is given in (50). We emphasize that the decay coupling constants ${}^{j_a}F^{j_1\,j_c}_{\lambda_1\,\lambda_c}$, ${}^{j_b}F^{j_2\,j_d}_{\lambda_2\,\lambda_d}$ and ${}^{j}F^{j_c\,j_d}_{\lambda_c\,\lambda_d}$ are unknown; so explicit expressions for the spin-density matrix cannot be calculated. However, a great deal of insight could be gained by studying the matrix for assumed values of F's as a function of t_c, t_d and ϕ, as given in (56).

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Appendices:

A Boreskov Treatment

Here we retrace a few essential elements of a paper by Boreskov[2]. In the overall CM system, i.e. the rest frame of $p_a + p_b$, he sets up a coordinate system with the z-axis along $-\vec{p}_3$ and assigns the vector $\vec{p}_a$ to lie in the (xz)-plane. In this frame, the vector $\vec{p}_1$ has the polar angles (ϑ_2, ϕ). In summary, we have

$$\begin{aligned} &\vec{p}_a = p_a(\vartheta_1, 0) \qquad &&\vec{p}_b = p_a(\pi - \vartheta_1, \pi) \\ &\vec{p}_3 = p_3(\pi, 0) \qquad &&\vec{p}_1 = p_1(\vartheta_2, \varphi) \qquad &&\vec{p}_2 = -\vec{p}_3 - \vec{p}_1 \end{aligned} \tag{A.1}$$

The plane formed by $\vec{p}_1$, $\vec{p}_3$ and $\vec{p}_2$ can be thought of as the (xz)-plane rotated by φ around the z-axis. Our coordinate system is very different from this setup. He writes down the production amplitude as follows:

$$\begin{aligned} \mathcal{T} &= \sum_{j_1\, j_2 m_2} d^{j_1}_{m_1 \Lambda_1}(\vartheta_1) D^{j_2\,*}_{m_2\,\Lambda_2}(\varphi, \vartheta_2, 0)\, T(j_1; j_2) \\ &= \sum_{j_1\, j_2 m_2} d^{j_1}_{m_1 \Lambda_1}(\vartheta_1)\, e^{i\, m_2 \varphi}\, d^{j_2}_{m_2\,\Lambda_2}(\vartheta_2)\, T(j_1; j_2) \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} \cos\vartheta_1 &= \frac{(t_d - m_2^2 - t_c)(t_c + m_1^2 - m_a^2) - 2t_c(s_{13} - m_3^2 - m_1^2)}{\Delta(t_c, t_d, m_3^2)\Delta(t_c, m_a^2, m_1^2)} \\ \cos\vartheta_2 &= \frac{(t_c - m_3^2 - t_d)(t_c + m_2^2 - m_b^2) - 2t_d(s_{23} - m_2^2 - m_3^2)}{\Delta(t_c, t_d, m_3^2)\Delta(t_d, m_b^2, m_2^2)} \\ \Delta(x, y, z) &= (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)^{1/2} \end{aligned} \tag{A.3}$$

(simply transcribed from the Boreskov paper—not checked independently).

The high-energy limit (the Regge picture) is attained by setting $\cos(\vartheta_1) \to \infty$ and $\cos(\vartheta_2) \to \infty$ which lead to the conidtions

$$\begin{aligned}
-t_c \gg -t_{c\,\min} &= -\frac{2(m_1^2-m_a^2)(m_3^2-t_d)}{s_{13}}, \quad m_a \neq m_1 \\
&= -m_e\,\frac{(m_3^2-t_d)^2}{s_{13}^2}, \quad m_e = m_a = m_1 \\
-t_d \gg -t_{d\,\min} &= -\frac{2(m_2^2-m_b^2)(m_3^2-t_c)}{s_{23}}, \quad m_b \neq m_2 \\
&= -m_f\,\frac{(m_3^2-t_c)^2}{s_{23}^2}, \quad m_f = m_b = m_2
\end{aligned} \tag{A.4}$$

(simply transcribed from the Boreskov paper—not checked independently).

We note here that the angles in $\cos\vartheta_1$ and $\cos\vartheta_2$ are very different from those which appeared in Section 3.

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