

Quantum Numbers for Hybrid Mesons in the Flux-tube Model —Version III—

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abstract

A detailed explanation is given on the transformation properties of the operators corresponding to the intrinsic parity P and the charge-conjugation C , acting on a $q\bar{q} + g$ wave function in the flux-tube model.

The formalism thus developed for the flux-tube model is applied to the first and second vibrational modes of the flux tube. The hierarchy of $J^{PC} = 1^{-+}$ exotic mesons, predicted in the flux-tube model, as well as an intriguing role played by $J^{PC} = 0^{--}$ exotic mesons, is presented.

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1 Introduction

The flux-tube model, pioneered by Isgur and Paton[1], has become the most successful phenomenological model for the mesons which contain gluonic excitations. The purpose of this note is to elaborate on certain aspects of their model to make it more accessible to experimental physicists—like me.

One of the key innovations of this note is to introduce two quantum numbers for the string excitations, the phonon number N which governs the string energy—and hence the mass shift due to the valence gluon—and the string helicity Λ (which they have already introduced). It is argued that a ket state corresponding to the string excitation could be denoted $|N \Lambda\rangle$, with little loss of information, in the phenomenological approach suitable for experimentalists.

From a host of exotic mesons predicted in the flux-tube model, we concentrate on two types— $J^{PC} = 1^{-+}$ and $J^{PC} = 0^{- -}$ —and show that the second mode of string excitation leads naturally to the two types being produced with their mass close to each other but very high, probably in 3.8–4.2 GeV range.

2 Conventional Mesons and Beyond

Let L and S be the orbital angular momentum and the total intrinsic spin of a $q\bar{q}$ system. Let s_1 and s_2 be the spins of q and \bar{q} , i.e. $s_1 = s_2 = 1/2$. Then, one readily finds

$$\vec{J} = \vec{L} + \vec{S} \quad \text{and} \quad \vec{S} = \vec{s}_1 + \vec{s}_2 \quad (1)$$

and

$$\begin{aligned} P &= (-)^{L+1} \\ C &= (-)^{L+S} \\ PC &= (-)^{S+1} \end{aligned} \quad (2)$$

and the wave function in a state of J , L and S is given by

$$\begin{aligned} |JMLS\rangle &= \sqrt{\frac{2L+1}{4\pi}} \sum_{\substack{m_1 m_2 \\ m_s m}} (s_1 m_1 s_2 m_2 | S m_s) (S m_s L M_L | J M) \\ &\quad \times \int d\Omega D_{M_L 0}^{L*}(\phi, \theta, 0) |\Omega, s_1 m_1 s_2 m_2\rangle \end{aligned} \quad (3)$$

where M is the z -component of spin J in a coordinate system given in the rest frame of $q\bar{q}$ and $\Omega = (\theta, \phi)$ specifies the direction of the breakup momentum in this rest frame. We use the notation $(j_1 m_1 j_2 m_2 | j m)$ to denote the usual Clebsch-Gordan coefficients. The formula given here is of course elementary; I will show later how this can be generalized to include a gluonic meson: $q\bar{q} + \text{gluon}$.

Given a set of relations (1), one finds that the following states are not allowed for a $q\bar{q}$ system, i.e

$$J^{PC} = 0^{--}, \quad 0^{+-}, \quad 1^{-+}, \quad 2^{+-}, \quad 3^{-+}, \dots \quad (4)$$

In this note the exotic mesons listed above are designated using the following notation:

$I^G(J^{PC})$	Exotics	$I^G(J^{PC})$	Exotics
$1^+(0^{--})$	ρ_0	$0^-(0^{--})$	ω_0
$1^+(0^{+-})$	b_0	$0^-(0^{+-})$	h_0
$1^-(1^{-+})$	π_1	$0^+(1^{-+})$	η_1
$1^+(2^{+-})$	b_2	$0^-(2^{+-})$	h_2
$1^-(3^{-+})$	π_3	$0^+(3^{-+})$	η_3

3 Hybrid mesons

In the flux-tube model pioneered by Isgur and Paton[1] and further developed by others[2], an excited gluon in $q\bar{q} + \text{gluon}$ is described by two transverse polarization states of a string, which may be taken to be clockwise and anticlockwise about the $q\bar{q}$ axis. Let $n_m^{(+)}$ [$n_m^{(-)}$] be the number of clockwise (anticlockwise) phonons in the m th mode of string excitation.^a Evidently, we must have $n_m^{(+)} = n_m^{(-)} = 0$ for $m = 0$, corresponding to the ground-state (or hidden) gluon string.^b

^a In the continuum limit, the transverse string displacement is proportional to $y_m(\xi) = \sin(m\pi\xi)$, where ξ goes from 0 to 1 along a straight line (the z -axis) connecting the two quarks.

^b This is not true in the flux-tube model (private communication, P. Page). Both ordinary quarkonia and gluonic hybrids are endowed with strings, and so m ranges from 1 to $m(\text{max})$ in all cases. However, we allow $m = 0$ to indicate a cross-over to the ‘conventional’ quark model with absence of strings.

Two important quantities are defined from these

$$\begin{aligned}\Lambda &= \sum_m [n_m^{(+)} - n_m^{(-)}] \\ N &= \sum_m m [n_m^{(+)} + n_m^{(-)}]\end{aligned}\tag{5}$$

It is clear from the definition that Λ is the helicity of the flux tube along the $q\bar{q}$ axis. The number N is a new quantity; it will be given the name ‘phonon number’—as it represents the sum of all the phonons in the problem weighted by the mode number m . The N should properly be called the ‘phonon energy number,’ because the string excitation energy is given by[1]

$$E(\text{string}) \propto \dots + N + \dots\tag{6}$$

in the continuum approximation.^c Note that, for the ground-state gluon string or $m = 0$, one has $\Lambda = N = 0$.

A hybrid state with momentum \hat{k} along the $q\bar{q}$ axis, defined in the $q\bar{q}$ rest frame, may be written

$$|\hat{k}; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle\tag{7}$$

It should be emphasized that a complete vibrational degrees of freedom of the string is specified by the set $\{n_m^{(+)}, n_m^{(-)}\}$ only, but it is convenient to introduce two additional quantities N and Λ in the specification of a ket state. N was not included as a member of the labels specifying a wave function in the original paper of Isgur and Paton[1]. However, N plays an important role as seen later, and thus—in this note—it has been included along with Λ . In the formula above, the quark labels i and j refer to the flavor. Thus, limiting oneself to light quarks only, one may set

$$\{q_1, q_2, q_3\} = \{u, d, s\}\tag{8}$$

Isgur and Paton[1] show that a state of total orbital angular momentum L with its z -component M_L is given by

$$\begin{aligned}|LM_L; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle &= \sqrt{\frac{2L+1}{4\pi}} \\ &\times \int d\Omega D_{M_L \Lambda}^{L*}(\phi, \theta, 0) |\Omega; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle\end{aligned}\tag{9}$$

^c This result is obtained by inserting Eq.(18) into Eq.(19) in Ref. [1]. Their Eq.(19) reduces Eq.(18) in the limit $m(\text{max}) \rightarrow \infty$ or $m \ll m(\text{max})$; this is the continuum limit.

where $\Omega(\theta, \phi)$ describes the direction \hat{k} . Note that this is very similar to the description of a two-body system in the helicity formalism[3].^d

The complete wave function is now given by

$$\begin{aligned} & |JM; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle \\ &= \sum_{M_L m_s} (LM_L Sm_s | JM) |LM_L Sm_s; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle \end{aligned} \quad (10)$$

where

$$\begin{aligned} & |LM_L Sm_s; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle \\ &= \sum_{m_1 m_2} (s_1 m_1 s_2 m_2 | Sm_s) |LM_L; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle \end{aligned} \quad (11)$$

and again $s_1 = s_2 = 1/2$ for q_i and \bar{q}_j . These formulas show that the counterpart of the wave function (3) in the case of gluonic hybrids is, suppressing the indices for quark flavors,

$$\begin{aligned} |JM L S \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle &= \sum_{\substack{m_1 m_2 \\ m_s M_L}} (s_1 m_1 s_2 m_2 | Sm_s) (Sm_s LM_L | JM) \\ &\times \int d\Omega D_{M_L \Lambda}^{L*}(\phi, \theta, 0) |\Omega, s_1 m_1; s_2 m_2; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle \end{aligned} \quad (12)$$

There are a set of five rotational invariants specifying a state in the flux-tube model, i.e. J , L , S , N and Λ .^e

Denoting by \mathbb{P} and \mathbb{C} the operators corresponding to parity and charge conjugation, the hybrid states of a given L and S transform according to

$$\begin{aligned} \mathbb{P} |Lm Sm_s; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle &= (-)^{L+\Lambda+1} |Lm Sm_s; q_i \bar{q}_j; \{n_m^{(-)}, n_m^{(+)}\} N, -\Lambda\rangle \\ \mathbb{C} |Lm Sm_s; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle &= (-)^{L+S+\Lambda+N} |Lm Sm_s; q_j \bar{q}_i; \{n_m^{(-)}, n_m^{(+)}\} N, -\Lambda\rangle \end{aligned} \quad (13)$$

One sees that, if $m = 0$ (no phonons) and $i = j$ (neutral nonstrange states), then the ket states are in eigenstate of \mathbb{P} and \mathbb{C} . The corresponding eigenvalues P and C are as given in (1), so that one has $PC = (-)^{S+1}$. The transformation law under the combined operation is

$$\mathbb{P} \mathbb{C} |Lm Sm_s; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle = (-)^{S+N+1} |Lm Sm_s; q_j \bar{q}_i; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda\rangle \quad (14)$$

^d The reader will note that our definition of the L state is different in two respects: 1) The Euler angles are $(\phi, \theta, 0)$ and not $(\phi, \theta, -\phi)$; 2) The rotation functions appear as D^* and not just D . It is seen that our definitions confirm to the convention adopted in Chung's CERN Yellow Report.

^e Strictly speaking, only the quantity $|\Lambda|$ is rotationally invariant (private communication, P. Page).

The corresponding eigenvalue reduces to the usual one for $N = 0$, as expected.

The hadrons, including hybrids, must come in eigenstates of \mathbb{P} and, for neutral nonstrange members, in eigenstates of \mathbb{C} . So the proper state of a hybrid must be written, with $\eta_0 = \pm 1$,

$$\begin{aligned} & |Lm Sm_s; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} \eta_0 N \Lambda \rangle \\ & = a_0 [|Lm Sm_s; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} N \Lambda \rangle + \eta_0 |Lm Sm_s; q_i \bar{q}_j; \{n_m^{(-)}, n_m^{(+)}\} N -\Lambda \rangle] \end{aligned} \quad (15)$$

where a_0 is the normalization constant given by $1/2$, if the first and the second terms above are equal, i.e.

$$\{n_m^{(+)}, n_m^{(-)}\} = \{n_m^{(-)}, n_m^{(+)}\} \quad (16)$$

(A necessary consequence of this is that $\Lambda = 0$.) and it is given by $1/\sqrt{2}$ otherwise. The actions of \mathbb{P} and \mathbb{C} on the new states are

$$\begin{aligned} & \mathbb{P} |Lm Sm_s; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} \eta_0 N \Lambda \rangle \\ & = \eta_0 (-)^{L+\Lambda+1} |Lm Sm_s; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} \eta_0 N \Lambda \rangle \\ & \mathbb{C} |Lm Sm_s; q_i \bar{q}_j; \{n_m^{(+)}, n_m^{(-)}\} \eta_0 N \Lambda \rangle \\ & = \eta_0 (-)^{L+S+\Lambda+N} |Lm Sm_s; q_j \bar{q}_i; \{n_m^{(+)}, n_m^{(-)}\} \eta_0 N \Lambda \rangle \end{aligned} \quad (17)$$

and the new eigenvalues are

$$\begin{aligned} P & = \eta_0 (-)^{L+\Lambda+1} \\ C & = \eta_0 (-)^{L+S+\Lambda+N} \\ PC & = (-)^{S+N+1} \end{aligned} \quad (18)$$

It clear that, from (5), (6) and (13), one must set $\eta_0 = +1$ if $a_0 = 1/2$.^f If in addition $N = 0$ (or $m = 0$), then the formula above becomes identical to (2). The presence of $\eta_0 = \pm 1$ in (18) leads to a decoupling of P and C from the internal quantum numbers such as L and S if $\Lambda = 0$; only the combined quantity PC retain a link to the total intrinsic spin $S = 0, 1$ and the phonon number N .

The lowest-order gluonic hybrid occurs evidently for one $m = 1$ ‘clockwise’ phonon. In this case, $n_1^{(+)} = 1$ and $n_1^{(-)} = 0$, so that $\Lambda = N = 1$ and $PC = (-)^S$. Note that, from (9), $L = 0$ is not allowed; so one may set $L = 1$ as the first nontrivial example. The following table summaries this case:

^f This condition is met if a ket state is already in an eigenstate of \mathbb{P} and \mathbb{C} without the symmetrization of (15). Evidently, $\Lambda = 0$ is a necessary (but not sufficient) condition (private communication, P. Page). See also Table V.

Table I: $n_1^{(+)} = 1$ and $n_1^{(-)} = 0$; $N = 1$ and $\Lambda = 1$

L	S	$^{2S+1}L_J(q\bar{q})$	$J^{PC}(q\bar{q})$	$J^{PC}(q\bar{q} + g)$
1	0	1P_1	1^{+-}	1^{++} 1^{--}
1	1	3P_J	$0^{++}, 1^{++}, 2^{++}$	$0^{-+}, 1^{-+}, 2^{-+}$ $0^{+-}, 1^{+-}, 2^{+-}$
2	0	1D_2	2^{-+}	2^{++} 2^{--}
2	1	3D_J	$1^{--}, 2^{--}, 3^{--}$	$1^{-+}, 2^{-+}, 3^{-+}$ $1^{+-}, 2^{+-}, 3^{+-}$

Thus the flux-tube model predicts, in the lowest-order gluonic excitation, a series of the PC -doublets which must have exactly the same mass. For example, observation of an exotic 1^{-+} gluonic hybrid entails existence of a non-exotic 1^{+-} gluonic hybrid at exactly the same mass[4]. The isovector and the isoscalar states for the J^{PC} -exotic gluonic hybrids ($q\bar{q} + g$) for one $m = 1$ phonon with $L = 1$ —for the case of the table above—are predicted to have the masses, with $n = \{u, d\}$,

Constituents	J^{PC} -Exotics	Mass (GeV)
$n\bar{n} + \text{gluon}$	$b_0, h_0, \pi_1, \eta_1, b_2, h_2$	~ 1.9
$s\bar{s} + \text{gluon}$	$b_{s0}, h_{s0}, \pi_{s1}, \eta_{s1}, b_{s2}, h_{s2}$	~ 2.1
$c\bar{c} + \text{gluon}$	$b_{c0}, h_{c0}, \pi_{c1}, \eta_{c1}, b_{c2}, h_{c2}$	~ 4.3
$b\bar{b} + \text{gluon}$	$b_{b0}, h_{b0}, \pi_{b1}, \eta_{b1}, b_{b2}, h_{b2}$	~ 10.8

Other phonon excitations of interest are given below in tabular forms:

 Table II: $n_1^{(+)} = 1$ and $n_1^{(-)} = 1$; $N = 2$ and $\Lambda = 0$

L	S	$^{2S+1}L_J(q\bar{q})$	$J^{PC}(q\bar{q})$	$J^{PC}(q\bar{q} + g)$
0	0	1S_0	0^{-+}	0^{-+}
0	1	3S_1	1^{--}	1^{--}
1	0	1P_1	1^{+-}	1^{+-}
1	1	3P_J	$0^{++}, 1^{++}, 2^{++}$	$0^{++}, 1^{++}, 2^{++}$

So this case consists only of non-exotic J^{PC} 's, although the string excitations lead to higher masses [see (6)]. Consider the following case:

Table III: $n_1^{(+)} = 2$ and $n_1^{(-)} = 0$; $N = 2$ and $\Lambda = 2$

L	S	$^{2S+1}L_J(q\bar{q})$	$J^{PC}(q\bar{q})$	$J^{PC}(q\bar{q} + g)$
2	0	1D_2	2^{-+}	2^{-+} 2^{+-}
2	1	3D_J	$1^{--}, 2^{--}, 3^{--}$	$1^{--}, 2^{--}, 3^{--}$ $1^{++}, 2^{++}, 3^{++}$

which gives one exotic $J^{PC} = 2^{+-}$. Next, one may consider the case with $m = 2$

Table IV: $n_2^{(+)} = 1$ and $n_2^{(-)} = 0$; $N = 2$ and $\Lambda = 1$

L	S	$^{2S+1}L_J(q\bar{q})$	$J^{PC}(q\bar{q})$	$J^{PC}(q\bar{q} + g)$
1	0	1P_1	1^{+-}	1^{+-} 1^{-+}
1	1	3P_J	$0^{++}, 1^{++}, 2^{++}$	$0^{++}, 1^{++}, 2^{++}$ $0^{--}, 1^{--}, 2^{--}$

This gives two exotic J^{PC} 's, i.e. $J^{PC} = 1^{-+}$ and, for the first time, a spin-zero exotic $J^{PC} = 0^{--}$. As a final example, consider a case in which $m = 1$ and $m = 2$ are mixed

Table V: $n_2^{(+)} = 1$ and $n_1^{(-)} = 1$; $N = 3$ and $\Lambda = 0$

L	S	$^{2S+1}L_J(q\bar{q})$	$J^{PC}(q\bar{q})$	$J^{PC}(q\bar{q} + g)$
0	0	1S_0	0^{-+}	0^{++} 0^{--}
0	1	3S_1	1^{--}	1^{+-} 1^{-+}
1	0	1P_1	1^{+-}	1^{++} 1^{--}
1	1	3P_J	$0^{++}, 1^{++}, 2^{++}$	$0^{-+}, 1^{-+}, 2^{-+}$ $0^{+-}, 1^{+-}, 2^{+-}$

This is the first example for which $\Lambda = 0$ and yet the condition (16) is not satisfied, leading to a PC -doubling. Again, for the first time, $N = 3$ is encountered, so the mass should be even higher than the $N = 2$ case.

So far one has encountered a set of three $J^{PC} = 1^{-+}$ exotic mesons. Two of these exotic occur for one $m = 1$ phonon as shown in Table I. The ground state clearly corresponds to the P -wave state, with the D -wave exotic meson occurring a higher mass. The P -wave exotic state in Table IV, corresponding to one $m = 2$ phonon excitation, must have a mass higher than the ground state of Table I; its mass is considered to be greater[5] than that of the D -wave mesons of Table I, and it must be accompanied by a $J^{PC} = 0^{--}$ meson with its mass nearby.

4 Conclusions

In the flux-tube model, the string excitations contribute to the characteristics of the resulting mesons through two quantum numbers; the string helicity Λ along the $q\bar{q}$ axis and the phonon number N which fixes the energy of the string excitations. They have a major effect—indeed—on the internal makeup and the external characteristics of the hybrid mesons: one must have a PC -doubling of the resulting meson spectra under certain conditions (even when $\Lambda = 0$; see Table V, for example) and, furthermore, one must have $L \geq |\Lambda|$; and N enters in the formula for the sign of PC and that for the mass.

The ground-state $J^{PC} = 1^{-+}$ exotic meson corresponds to one $m = 1$ phonon excitation with the P -wave orbital angular momentum between q and \bar{q} . The next $J^{PC} = 1^{-+}$ exotic meson, at a higher mass, corresponds to the D -wave state. The flux-tube model predicts another pair of $J^{PC} = 1^{-+}$ exotic meson resulting from three $m = 1$ phonons, e.g. $n_1^{(+)} = 2$, $n_1^{(-)} = 1$, $\Lambda = 1$ and $N = 3$, but their masses must be higher. It is clear that this process continues with any odd number of $m = 1$ phonons such that $\Lambda = \pm 1$ and $N = \text{odd}$. The model predicts, in addition, a series of $J^{PC} = 1^{-+}$ exotics arising from any odd number of $m = 2$ phonon excitations such that $\Lambda = \pm 1$ and $N = \text{even}$. But their characteristics is very different; instead of a pair of $J^{PC} = 1^{-+}$ exotics, one $J^{PC} = 1^{-+}$ exotic meson is accompanied by one $J^{PC} = 0^{--}$ exotic meson, with their mass very close to each other. Other more complicated phonon structure is also possible; Table V shows an example of two

phonons of mixed m , which gives rise to two S -wave $J^{PC} = 0^{--}$ and $J^{PC} = 1^{-+}$ exotic mesons.

According to the calculations of the QCD sum rule by Latorre *et al.*[6], a $J^{PC} = 0^{--}$ ($n\bar{n} + g$) state should occur with mass in the range 3.8–4.2 GeV, well into the mass range of charmonia. So, one may speculate that $J^{PC} = 1^{-+}$ exotic mesons, arising from $m = 2$ phonons, must occur with their mass some 2.0 GeV above those resulting from $m = 1$ phonons. This is in sharp contrast to the mass difference expected to occur between the $J^{PC} = 1^{-+}$ exotic mesons with $L = 1$ and $L = 2$ in the lowest-order string excitation. According to P. Page, the $L = 2$ states are 400 MeV heavier than the $L = 1$ states for light quarks and 270 MeV heavier for $c\bar{c}$ states.

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