BNL PREPRINT

BNL-QGS-02-011

# Two-Photon Amplitudes of Quarkonia in Helicity Formalism

## —Version II—


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January 18, 2002


## abstract

We work out in some detail the amplitudes for a quarkonium $(^{2S+1}L_J)$ coupling to two photons. For this purpose we start with the amplitude obtained by applying the standard QED prescription to the diagrams which involve $t$-channel exchange of a quark in the process $q + \bar{q} \rightarrow \gamma + \gamma$. The resulting nonrelativistic amplitudes in helicity formulation show why helicity-two states for tensor mesons tend to dominate over helicity-zero.



* under contract number DE-AC02-98CH10886 with the U.S. Department of Energy

# 1 Introduction

The main purpose of this note is to work out the amplitudes for a quarkonium ($^{2S+1}L_J$) coupling to two photons. The reader may consult several papers in which various aspects of this problem has been dealt with—see Ackleh and Barnes[1], Anderson, Austern and Cahn[2], and Li, Close and Barnes[3]. For a general result on two-photon helicity selection rules, the reader is referred to Ackleh, Barnes and Close[4]. For an earlier work on this topic, one must cite a paper by Grassberger and Kögerler[5], who pointed out dominance of helicity-two states for tensor mesons, from the general principles as applied to two-photon elastic scattering mediated by quarkonia in the $s$-channel.

The appropriate diagram for a quarkonium ($^{2S+1}L_J$) coupling to two photons is given in Fig. 1. It is seen that the spin states for $q$ and $\bar{q}$ constitute intermediate states and are

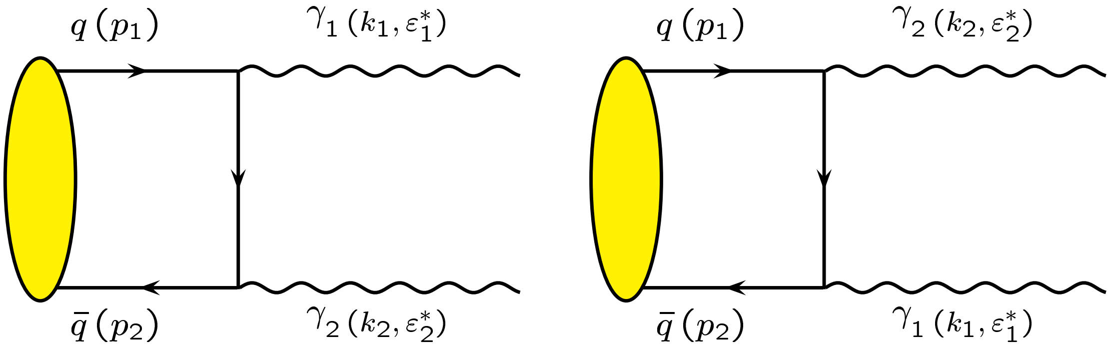


Figure 1: Diagrams for $q + \bar{q} \to \gamma + \gamma$. The blobs on the left represent a $q\bar{q}$ initial state in a given $|LSJM\rangle$.

thus summed over in an amplitude. For this reason, we find the standard trace techniques involving $\gamma_\mu$ matrices ill-suited for the problem; instead we work out the amplitudes in the Pauli spinor basis. The photon helicities specify the final states, so that they are summed over *only after* taking the absolute square of the amplitudes. Therefore, the photon helicicities may be considered 'external variables' in evaluating the amplitudes.

In section 2, we start with the angular distributions resulting from the helicity and canonical formalisms[6]. Section 3 is devoted to a derivation of the appropriate amplitudes in the spinor basis. The process of projecting out the nonrelativistic amplitudes into a given state $|LSJM\rangle$ is detailed in Section 4. The resulting amplitudes are cast into a form which can be easily applied to arbitrary $L$. In Section 5, the results are applied to $S$-, $P$-, $D$- and $F$-wave quarkonia. Concluding remarks are given in Section 6. In particular, we show that helicity-two dominance of tensor mesons in the nonrelativistic limit follows naturally from the structure of the helicity amplitudes, expressed as a function of the scattering angle for the process $q + \bar{q} \to \gamma + \gamma$. The problem of constructing amplitudes in a given $|Sm_s\rangle$ ($S = 0$ or 1) from $q$ and $\bar{q}$ states is dealt with in Appendix A. Relevant $d^J_{m'm}(\theta)$ functions and related functions are listed in Appendix B.

The reader will note that we have devoted a considerable amount of effort exhibiting important intermediate steps in the derivation, in order to facilitate verification of the results given in this note.

# 2 Preliminaries

We recall that a two-photon system is best described in the helicity formalism of Jacob and Wick[6]. The helicity-coupling amplitudes satisfy

$$F^J_{\lambda_1\lambda_2} = P(-)^J\, F^J_{-\lambda_1 -\lambda_2}, \qquad F^J_{\lambda_1\lambda_2} = (-)^J\, F^J_{\lambda_2\lambda_1} \tag{1}$$

where $\lambda_1$ and $\lambda_2$ are the helicities of the two photons. The first relationship comes from parity conservation and the second from the requirement of Bose symmetry. We see that

$$\begin{aligned} F^J_{\lambda,\lambda} &= (-)^J\, F^J_{\lambda,\lambda} \\ F^J_{\lambda,-\lambda} &= P(-)^J\, F^J_{-\lambda,\lambda} = (-)^J\, F^J_{-\lambda,\lambda} \end{aligned} \tag{2}$$

Let $\Lambda = \lambda_1 - \lambda_2$ be the net helicity of the system. From (2), we find the nonzero amplitudes as given in Table I.

| Table I: Amplitudes for two-photon systems | | | | | |
|---|---|---|---|---|---|
| $J^{PC}$ | $\Lambda$ | Amplitude | $J^{PC}$ | $\Lambda$ | Amplitude |
| $0^{-+}$ | 0 | | $3^{-+}$ | 0 | 0 |
| $0^{-+}$ | 2 | 0 | $3^{-+}$ | 2 | 0 |
| $0^{++}$ | 0 | | $3^{++}$ | 0 | 0 |
| $0^{++}$ | 2 | 0 | $3^{++}$ | 2 | |
| $1^{-+}$ | 0 | 0 | $4^{-+}$ | 0 | |
| $1^{-+}$ | 2 | 0 | $4^{-+}$ | 2 | 0 |
| $1^{++}$ | 0 | 0 | $4^{++}$ | 0 | |
| $1^{++}$ | 2 | 0 | $4^{++}$ | 2 | |
| $2^{-+}$ | 0 | | $5^{-+}$ | 0 | 0 |
| $2^{-+}$ | 2 | 0 | $5^{-+}$ | 2 | 0 |
| $2^{++}$ | 0 | | $5^{++}$ | 0 | 0 |
| $2^{++}$ | 2 | | $5^{++}$ | 2 | |

Note that $F^J_{\lambda,\lambda} = 0$ if $J$ is odd and that $F^J_{\lambda,-\lambda} = 0$ if $P$ is odd, so that $J^{PC} = \text{odd}^{-+}$ states do not couple to two photons. In addition, the amplitudes for $J^{PC} = \text{odd}^{++}$ states is zero for $\Lambda = 0$, since $F^J_{\lambda,\lambda} = 0$ for odd $J$. Finally, the amplitude for $0^{++}$ or $1^{++}$ must be zero for $\Lambda = 2$ (not allowed for $J = 0$ or 1). We see that $J = 1$ states do not couple to two photons—this is the Landau-Yang theorem.

A quarkonium coupling to two photons must have $C = (-)^{S+L} = +$, so we must have $S+L = \text{even}$ and $P = (-)^{L+1}$, where $S$ is the total intrinsic spin and $L$ is the orbital angular momentum between $q$ and $\bar{q}$. We must have either

$$S = 0,\ L = \text{even},\ P = \text{odd and } J = L \qquad \text{or}$$
$$S = 1,\ L = \text{odd},\ P = \text{even and } J = L-1,\ J = L,\ J = L+1.$$

We see, in addition, that a state with $J^{PC} = \text{odd}^{-+}$ cannot couple to a $(q+\bar{q})$; hence it is exotic.

We turn to a review of a few essential features of canonical and helicity formalisms[6]. Consider a system of two massive particles with spins $s_1$ and $s_2$, helcities $\mu_1$ and $\mu_2$, and canonical $z$-components of spin $m_1$ and $m_2$. Let $\Omega = (\theta, \phi)$ describe the orientation of particle 1 in the rest frame. There are two ways of describing the two-particle system. In the helicity formulation, one has

$$|\mu_1\mu_2\, JM\rangle = N_J \int \mathrm{d}\Omega\, D^{J\,*}_{M\,\mu}(\phi,\theta,0)\,|\Omega\,\mu_1\mu_2\rangle \tag{3}$$

where

$$N_J = \left(\frac{2J+1}{4\pi}\right)^{1/2} \tag{4}$$

and $\mu = \mu_1 - \mu_2$. In the canonical prescription, two-particle system is described by

$$\begin{aligned} |LS\, JM\rangle = &\sum_{m_1 m_2} (s_1 m_1\, s_2 m_2|S m_s)(S m_s\, Lm|JM) \\ &\times \int \mathrm{d}\Omega\, Y^m_L(\Omega)\,|\Omega\, m_1 m_2\rangle \end{aligned} \tag{5}$$

Transformation laws under parity operation $\Pi$ are

$$\Pi\,|\Omega\,\mu_1\mu_2\rangle = \eta_1\eta_2(-)^{-s_1-s_2}\,|\bar\Omega\, -\mu_1 - \mu_2\rangle, \quad \Pi\,|\Omega\, m_1 m_2\rangle = \eta_1\eta_2\,|\bar\Omega\, m_1 m_2\rangle \tag{6}$$

where $\bar\Omega = (\pi-\theta, \pi+\phi)$, and $\eta_1$ and $\eta_2$ are intrinsic parities of particles 1 and 2, so that

$$\Pi\,|\mu_1\mu_2\, JM\rangle = \eta_1\eta_2(-)^{J-s_1-s_2}\,|-\mu_1-\mu_2\, JM\rangle, \quad \Pi\,|LS\, JM\rangle = \eta_1\eta_2(-)^L\,|LS\, JM\rangle \tag{7}$$

The two ket states are related by

$$\begin{aligned} |\mu_1\mu_2\, JM\rangle &= \sum_{LS} \left(\frac{2L+1}{2J+1}\right)^{1/2} (s_1\mu_1\, s_2 -\mu_2|S\mu)(S\mu\, L0|J\mu)\,|LS\, JM\rangle \\ |LS\, JM\rangle &= \left(\frac{2L+1}{2J+1}\right)^{1/2} \sum_{\mu_1\mu_2} (s_1\mu_1\, s_2 -\mu_2|S\mu)(S\mu\, L0|J\mu)\,|\mu_1\mu_2\, JM\rangle \end{aligned} \tag{8}$$

We see that

$$|LS\, JM\rangle = N_L \sum_{\mu_1\mu_2} (s_1\mu_1\, s_2 -\mu_2|S\mu)(S\mu\, L0|J\mu) \int \mathrm{d}\Omega\, D^{J\,*}_{M\,\mu}(\phi,\theta,0)\,|\Omega\,\mu_1\mu_2\rangle \tag{9}$$

We are now ready to write down the amplitudes for the process $q+\bar q \to \gamma+\gamma$ in the helicity formalism. For this purpose, fix the coordinate system such that the $z$-axis is along the momentum of the two photons. The $z$-component of spin of the two-photon system can take on $\Lambda = 0$ or $\Lambda = 2$, since the orbital angular momentum between two photons cannot have its spin projection along the $z$-axis. Let $\Omega(\theta,\phi)$ describe the orientation of the $q\bar q$ axis in its rest frame. The helicity transition amplitude for the process $q+\bar q \to \gamma+\gamma$ can be written

$$H^{\mu_1\mu_2}_{\Lambda}(\Omega) = \langle \Omega_0\, \lambda_1\lambda_2|T|\Omega\,\mu_1\mu_2\rangle \tag{10}$$

where $\Lambda = \lambda_1 - \lambda_2$ and $\Omega_0 = (0,0)$. Parity conservation in the process demands that

$$\langle \Omega_0 \, \lambda_1 \lambda_2 | T | \Omega \, \mu_1 \mu_2 \rangle = \langle \bar{\Omega}_0 \, - \lambda_1 - \lambda_2 | T | \bar{\Omega} \, - \mu_1 - \mu_2 \rangle \tag{11}$$

so that

$$H_\Lambda^{\mu_1 \mu_2}(\Omega) = H_{-\Lambda}^{-\mu_1 - \mu_2}(\Omega) \tag{12}$$

Then the amplitude corresponding to a state $|LSJ\Lambda\rangle$ can be written, from (9),

$$\begin{aligned} A_\Lambda({}^{2S+1}L_J) = N_L \sum_{\mu_1 \mu_2} & (s_1 \mu_1 s_2 - \mu_2 | S\mu) \, (S\mu \, L0 | J\mu) \\ & \times \int \mathrm{d}\Omega \, D_{\Lambda\mu}^{J\,*}(\phi, \theta, 0) \, H_\Lambda^{\mu_1 \mu_2}(\Omega) \end{aligned} \tag{13}$$

so that

$$\begin{aligned} A_\Lambda({}^1L_L) &= \frac{1}{\sqrt{2}} N_L \int \mathrm{d}\Omega \, D_{\Lambda\,0}^{L\,*}(\phi,\theta,0) \, \left[ H_\Lambda^{(++)}(\Omega) - H_\Lambda^{(--)}(\Omega) \right] \\ A_\Lambda({}^3L_J) &= \frac{1}{\sqrt{2}} N_L \, (10\, L0 | J0) \int \mathrm{d}\Omega \, D_{\Lambda\,0}^{J\,*}(\phi,\theta,0) \, \left[ H_\Lambda^{(++)}(\Omega) + H_\Lambda^{(--)}(\Omega) \right] \\ & \quad + N_L \, (11\, L0 | J1) \int \mathrm{d}\Omega \, \left[ D_{\Lambda\,1}^{J\,*}(\phi,\theta,0) \, H_\Lambda^{(+-)}(\Omega) \right. \\ & \qquad \left. + (-)^J \, D_{\Lambda\,-1}^{J\,*}(\phi,\theta,0) \, H_\Lambda^{(-+)}(\Omega) \right] \end{aligned} \tag{14}$$

We now turn to the amplitudes for $q\bar{q}$ in the canonical description. Photons remain in the helicity domain, as they are massless. The relevant amplitude can be written

$$A_\Lambda^{m_1 m_2}(\Omega) = \langle \Omega_0 \, \lambda_1 \lambda_2 | T | \Omega \, m_1 m_2 \rangle \tag{15}$$

The amplitude (13) reads, from (5),

$$A_\Lambda({}^{2S+1}L_J) = \sum_{m_1 m_2} (s_1 m_1 \, s_2 m_2 | S m_s)(S m_s \, Lm | J\Lambda) \int \mathrm{d}\Omega \, Y_L^m(\Omega) \, A_\Lambda^{m_2 m_1}(\Omega) \tag{16}$$

where $|s_1 m_1\rangle$ refers to $q$ and $|s_2 m_2\rangle$ to $\bar{q}$. Noting that

$$Y_L^m(\Omega) = N_L \, d_{m0}^L(\theta) \, \exp[m \, i \, \phi] \tag{17}$$

the amplitudes take on the form

$$
\begin{aligned}
A_0(^1L_L) &= \frac{1}{\sqrt{2}} N_L \int \mathrm{d}\Omega\, d^L_{00}(\theta)\, \left[A_0^{(-+)}(\Omega) - A_0^{(+-)}(\Omega)\right], \quad A_2(^1L_L) = 0 \\
A_0(^3L_J) &= \frac{1}{\sqrt{2}} N_L\, (10L0|J0) \int \mathrm{d}\Omega\, d^L_{00}(\theta)\, \left[A_0^{(+-)}(\Omega) + A_0^{(-+)}(\Omega)\right] \\
&\quad + N_L\, (1\ {-1}\, L1|J0) \int \mathrm{d}\Omega\, d^L_{10}(\theta)\, \left[-(-)^J\, \mathrm{e}^{-i\,\phi}\, A_0^{(++)}(\Omega) + \mathrm{e}^{i\,\phi}\, A_0^{(--)}(\Omega)\right] \\
A_2(^3L_J) &= \frac{1}{\sqrt{2}} N_L\, (10L2|J2) \int \mathrm{d}\Omega\, d^L_{20}(\theta)\, \mathrm{e}^{2\,i\,\phi} \left[A_2^{(+-)}(\Omega) + A_2^{(-+)}(\Omega)\right] \\
&\quad + N_L\, (11\, L1|J2) \int \mathrm{d}\Omega\, d^L_{10}(\theta)\, \mathrm{e}^{i\,\phi}\, A_2^{(++)}(\Omega) \\
&\quad + N_L\, (1\ {-1}\, L3|J2) \int \mathrm{d}\Omega\, d^L_{30}(\theta)\, \mathrm{e}^{3\,i\,\phi}\, A_2^{(--)}(\Omega)
\end{aligned}
\tag{18}
$$

Note that the $d$-functions in the canonical formalism reflect the orbital angular momentum $L$, whereas those in the helicity description reflect the spin $J$ directly. In addition, the net helicity $\Lambda$ in the helicity formalism appear in the $D$-functions as their first subscript and—in particular—the Clebsch-Gordan coefficients in the formlism do not depend on it.

It is seen that one needs a model to calculate either the $H(\Omega)$'s and $A(\Omega)$'s. To go further, therefore, we must introduce a specific mechanism for the process $q + \bar{q} \to \gamma + \gamma$. This is detailed in the following sections.

# 3 Amplitudes for $q + \bar{q} \to \gamma + \gamma$

The amplitude for this process (see Fig. 1) can be written, according to the Feynmann rules[7],

$$
\begin{aligned}
-i\,\mathcal{M} &= \bar{v}(\vec{p}_2)\left[(iQ\,\not\epsilon_1^*)\frac{i}{\not p_1 - \not k_1 - m_q}(iQ\,\not\epsilon_2^*) + (iQ\,\not\epsilon_2^*)\frac{i}{\not p_1 - \not k_2 - m_q}(iQ\,\not\epsilon_1^*)\right] u(\vec{p}_1) \\
&= -i\,\bar{v}(\vec{p}_2)\left[(Q\,\not\epsilon_1^*)\frac{\not p_1 - \not k_1 + m_q}{(p_1-k_1)^2 - m_q^2}(Q\,\not\epsilon_2^*) + (Q\,\not\epsilon_2^*)\frac{\not p_1 - \not k_2 + m_q}{(p_1-k_2)^2 - m_q^2}(Q\,\not\epsilon_1^*)\right] u(\vec{p}_1)
\end{aligned}
\tag{19}
$$

where $p_1 + p_2 = k_1 + k_2$. Adopting one *particular* convention for the charge-conjugation operator[8] (see Appendix A), we may write

$$\begin{aligned} u(\vec{p}_1) &= \left(\frac{E_q + m_q}{2m_q}\right)^{1/2} \begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}_1}{E_q + m_q} \end{pmatrix} \chi \\ v(\vec{p}_2) &= \left(\frac{E_{\bar{q}} + m_q}{2m_q}\right)^{1/2} \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}_2}{E_{\bar{q}} + m_q} \\ 1 \end{pmatrix} (-i\,\sigma_2)\,\chi \end{aligned} \tag{20}$$

so that

$$\bar{v}(\vec{p}_2) = v^\dagger(\vec{p}_2)\,\gamma_0 = \left(\frac{E_{\bar{q}} + m_q}{2m_q}\right)^{1/2} \chi^\dagger\,(i\,\sigma_2) \left(\frac{\vec{\sigma}\cdot\vec{p}_2}{E_{\bar{q}} + m_q}, -1\right) \tag{21}$$

where $Q$ is the charge of $q$ (hence $-Q$ is the charge of $\bar{q}$) and $m = m_{\bar{q}} = m_q$. We use the Coulomb gauge for the polarization vectors, in which $\varepsilon_1^0 = 0$ and $\varepsilon_2^0 = 0$. So the transversality conditions reduce to

$$\vec{k}_1 \cdot \vec{\varepsilon}_1^{\,*} = 0 \qquad \text{and} \qquad \vec{k}_2 \cdot \vec{\varepsilon}_2^{\,*} = 0 \tag{22}$$

In the center-of-mass frame, we have $\vec{p}_1 = -\vec{p}_2$, $2\vec{p} = \vec{p}_1 - \vec{p}_2 = 2\vec{p}_1$ and $E = E_1 = E_2 = \sqrt{\bar{p}^2 + m^2}$ where $\bar{p}^2 = (\vec{p}\cdot\vec{p})$. In addition, we have $\vec{k} = \vec{k}_1 = -\vec{k}_2$ and $k_1^0 = k_2^0 = \bar{k}$ where, again, $\bar{k}^2 = (\vec{k}\cdot\vec{k}\,)$. We also have $\sqrt{s} = M = 2E = 2\bar{k}$, so that $\bar{k} = E = \sqrt{\bar{p}^2 + m^2}$. Both $\bar{p}$ and $\bar{k}$ stand for the magnitude of the 3-momenta. In the rest frame we have

$$\begin{aligned} (p_1 - k_1)^2 - m_q^2 &= -2(p_1\cdot k_1) = -2E^2 + 2(\vec{p}\cdot\vec{k}) \equiv -2K_- \\ (p_1 - k_2)^2 - m_q^2 &= -2(p_1\cdot k_2) = -2E^2 - 2(\vec{p}\cdot\vec{k}) \equiv -2K_+ \end{aligned} \tag{23}$$

and, in addition,

$$\not{\varepsilon}_i^{\,*} = \begin{pmatrix} 0 & -\vec{\sigma}\cdot\vec{\varepsilon}_i^{\,*} \\ \vec{\sigma}\cdot\vec{\varepsilon}_i^{\,*} & 0 \end{pmatrix}, \qquad \mathrm{i} = 1,2 \tag{24}$$

$$\not{k}_1 = \begin{pmatrix} E & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & -E \end{pmatrix}, \quad \not{k}_2 = \begin{pmatrix} E & \vec{\sigma}\cdot\vec{k} \\ -\vec{\sigma}\cdot\vec{k} & -E \end{pmatrix}, \quad \not{p}_1 = \begin{pmatrix} E & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E \end{pmatrix} \tag{25}$$

Utilizing the well-known relationship

$$\vec{\sigma}\cdot\vec{a}\;\vec{\sigma}\cdot\vec{b} = \vec{a}\cdot\vec{b} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \tag{26}$$

we see that

$$\not{\varepsilon}_1^{\,*}\not{\varepsilon}_2^{\,*} = \begin{pmatrix} -a_+ & 0 \\ 0 & -a_+ \end{pmatrix}, \qquad \not{\varepsilon}_2^{\,*}\not{\varepsilon}_1^{\,*} = \begin{pmatrix} -a_- & 0 \\ 0 & -a_- \end{pmatrix} \tag{27}$$

where

$$a_\pm = \vec{\varepsilon}_1^{\,*} \cdot \vec{\varepsilon}_2^{\,*} \pm i\,\vec{\sigma} \cdot (\varepsilon_1^* \times \vec{\varepsilon}_2^{\,*}) \tag{28}$$

We find, in addition, that

$$\not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2^* = \begin{pmatrix} E\,a_+ & -b_+ \\ +b_+ & -E\,a_+ \end{pmatrix}, \qquad \not{\varepsilon}_2^* \not{k}_2 \not{\varepsilon}_1^* = \begin{pmatrix} E\,a_- & -b_- \\ +b_- & -E\,a_- \end{pmatrix} \tag{29}$$

and

$$\not{\varepsilon}_1^* \not{p}_1 \not{\varepsilon}_2^* = \begin{pmatrix} E\,a_+ & +c_+ \\ -c_+ & -E\,a_+ \end{pmatrix}, \qquad \not{\varepsilon}_2^* \not{p}_1 \not{\varepsilon}_1^* = \begin{pmatrix} E\,a_- & +c_- \\ -c_- & -E\,a_- \end{pmatrix} \tag{30}$$

where

$$\begin{aligned} b_\pm &= \pm\vec{\sigma} \cdot k\ \vec{\varepsilon}_1^{\,*} \cdot \vec{\varepsilon}_2^{\,*} + i\,\vec{k} \cdot (\vec{\varepsilon}_1^{\,*} \times \vec{\varepsilon}_2^{\,*}) \\ c_\pm &= (\vec{\sigma} \cdot \vec{\varepsilon}_1^{\,*}\ \vec{p} \cdot \vec{\varepsilon}_2^{\,*} + \vec{p} \cdot \vec{\varepsilon}_1^{\,*}\ \vec{\sigma} \cdot \vec{\varepsilon}_2^{\,*}) - \vec{\sigma} \cdot \vec{p}\ \vec{\varepsilon}_1^{\,*} \cdot \vec{\varepsilon}_2^{\,*} \mp i\,\vec{p} \cdot (\vec{\varepsilon}_1^{\,*} \times \vec{\varepsilon}_2^{\,*}) \end{aligned} \tag{31}$$

Now the amplitude can be written

$$\mathcal{M} = Q^2\,\chi^\dagger\,(i\,\sigma_2)\,F\,\chi \tag{32}$$

where

$$F = \left(\frac{E+m}{2m}\right) \left(\frac{\vec{\sigma} \cdot \vec{p}}{E+m}, \quad 1\right) G \begin{pmatrix} 1 \\ \dfrac{\vec{\sigma} \cdot \vec{p}}{E+m} \end{pmatrix} \tag{33}$$

so that

$$\begin{aligned} G &= \not{\varepsilon}_1^* \left(\frac{\not{p}_1 - \not{k}_1 + m}{2K_-}\right) \not{\varepsilon}_2^* + \not{\varepsilon}_2^* \left(\frac{\not{p}_1 - \not{k}_2 + m}{2K_+}\right) \not{\varepsilon}_1^* \\ &= \frac{K_+}{2K_+K_-} \begin{pmatrix} -m\,a_+ & b_+ + c_+ \\ -b_+ - c_+ & -m\,a_+ \end{pmatrix} + \frac{K_-}{2K_+K_-} \begin{pmatrix} -m\,a_- & b_- + c_- \\ -b_- - c_- & -m\,a_- \end{pmatrix} \\ &= \frac{E^2}{K_+K_-} \begin{pmatrix} -e_+ & f_+ \\ -f_+ & -e_+ \end{pmatrix} + \frac{\vec{p} \cdot \vec{k}}{K_+K_-} \begin{pmatrix} -e_- & f_- \\ -f_- & -e_- \end{pmatrix} \end{aligned} \tag{34}$$

and

$$\begin{aligned} K_+K_- &= E^4 - (\vec{p} \cdot \vec{k})^2 \\ e_+ &= m\,\vec{\varepsilon}_1^{\,*} \cdot \vec{\varepsilon}_2^{\,*} \\ e_- &= i\,m\,\vec{\sigma} \cdot (\vec{\varepsilon}_1^{\,*} \times \vec{\varepsilon}_2^{\,*}) \\ f_+ &= -\vec{\sigma} \cdot \vec{p}\ \vec{\varepsilon}_1^{\,*} \cdot \vec{\varepsilon}_2^{\,*} + i\,\vec{k} \cdot (\vec{\varepsilon}_1^{\,*} \times \vec{\varepsilon}_2^{\,*}) + (\vec{\sigma} \cdot \vec{\varepsilon}_1^{\,*}\ \vec{p} \cdot \vec{\varepsilon}_2^{\,*} + \vec{p} \cdot \vec{\varepsilon}_1^{\,*}\ \vec{\sigma} \cdot \vec{\varepsilon}_2^{\,*}) \\ f_- &= \ \vec{\sigma} \cdot \vec{k}\ \vec{\varepsilon}_1^{\,*} \cdot \vec{\varepsilon}_2^{\,*} - i\,\vec{p} \cdot (\vec{\varepsilon}_1^{\,*} \times \vec{\varepsilon}_2^{\,*}) \end{aligned} \tag{35}$$

Note that $e_+$ and $f_+$ are symmetric under interchange of the two photons, whereas $e_-$ and $f_-$ are antisymmetric under the interchange.

We are now ready to exhibit the amplitude $F$ in its final form

$$F = \frac{E^2\, F_S + \vec{p}\cdot\vec{k}\, F_A}{E^4 - (\vec{p}\cdot\vec{k})^2} \tag{36}$$

where

$$\begin{aligned} F_S &= \left(\frac{E+m}{2m}\right)\left[-f_+ - \frac{\vec{\sigma}\cdot\vec{p}\, e_+ + e_+\, \vec{\sigma}\cdot\vec{p}}{E+m} + \frac{\vec{\sigma}\cdot\vec{p}\, f_+\, \vec{\sigma}\cdot\vec{p}}{(E+m)^2}\right] \\ F_A &= \left(\frac{E+m}{2m}\right)\left[-f_- - \frac{\vec{\sigma}\cdot\vec{p}\, e_- + e_-\, \vec{\sigma}\cdot\vec{p}}{E+m} + \frac{\vec{\sigma}\cdot\vec{p}\, f_-\, \vec{\sigma}\cdot\vec{p}}{(E+m)^2}\right] \end{aligned} \tag{37}$$

or

$$\begin{aligned} F_S = &- \left[\frac{E+m}{2m} - \frac{\bar{p}^2}{2m\,(E+m)}\right] i\,\vec{k}\cdot(\vec{\varepsilon}_1^{\,*}\times\vec{\varepsilon}_2^{\,*}) \\ &- \left[\frac{E+m}{2m} + \frac{\bar{p}^2}{2m\,(E+m)}\right] (\vec{\sigma}\cdot\vec{\varepsilon}_1^{\,*}\ \vec{p}\cdot\vec{\varepsilon}_2^{\,*} + \vec{p}\cdot\vec{\varepsilon}_1^{\,*}\ \vec{\sigma}\cdot\vec{\varepsilon}_2^{\,*}) \\ &+ \left[\frac{E+m}{2m} - \frac{\bar{p}^2}{2m\,(E+m)} - 1\right] \vec{\sigma}\cdot\vec{p}\ \vec{\varepsilon}_1^{\,*}\cdot\vec{\varepsilon}_2^{\,*} + \left[\frac{2}{m\,(E+m)}\right] \vec{\sigma}\cdot\vec{p}\ \vec{p}\cdot\vec{\varepsilon}_1^{\,*}\,\vec{p}\cdot\vec{\varepsilon}_2^{\,*} \\ F_A = &- \left[\frac{E+m}{2m} + \frac{\bar{p}^2}{2m\,(E+m)}\right] \vec{\sigma}\cdot\vec{k}\ \vec{\varepsilon}_1^{\,*}\cdot\vec{\varepsilon}_2^{\,*} \\ &+ \left[\frac{E+m}{2m} - \frac{\bar{p}^2}{2m\,(E+m)} - 1\right] i\,\vec{p}\cdot(\vec{\varepsilon}_1^{\,*}\times\vec{\varepsilon}_2^{\,*}) + \left[\frac{1}{m\,(E+m)}\right] \vec{\sigma}\cdot\vec{p}\ \vec{p}\cdot\vec{k}\ \vec{\varepsilon}_1^{\,*}\cdot\vec{\varepsilon}_2^{\,*} \end{aligned} \tag{38}$$

The coefficients in the brackets above can be significantly simplified in the nonrelativistic limit, i.e. $\beta \to 0$ where $\beta = \bar{p}/E$. Setting $m \simeq E(1-\beta^2/2)$ and dropping terms of order $\beta^4$ and higher, we write the the nonrelativistic amplitude as follows:

$$\begin{aligned} F_S^{\rm nr} &= -i\,\vec{k}\cdot(\vec{\varepsilon}_1^{\,*}\times\vec{\varepsilon}_2^{\,*}) - \left(1+\frac{\beta^2}{2}\right)(\vec{\sigma}\cdot\vec{\varepsilon}_1^{\,*}\ \vec{p}\cdot\vec{\varepsilon}_2^{\,*} + \vec{p}\cdot\vec{\varepsilon}_1^{\,*}\ \vec{\sigma}\cdot\vec{\varepsilon}_2^{\,*}) + \frac{1}{E^2}\,\vec{\sigma}\cdot\vec{p}\ \vec{p}\cdot\vec{\varepsilon}_1^{\,*}\,\vec{p}\cdot\vec{\varepsilon}_2^{\,*} \\ F_A^{\rm nr} &= -\left(1+\frac{\beta^2}{2}\right)\vec{\sigma}\cdot\vec{k}\ \vec{\varepsilon}_1^{\,*}\cdot\vec{\varepsilon}_2^{\,*} + \frac{1}{2E^2}\vec{\sigma}\cdot\vec{p}\ \vec{p}\cdot\vec{k}\ \vec{\varepsilon}_1^{\,*}\cdot\vec{\varepsilon}_2^{\,*} \end{aligned} \tag{39}$$

It is seen that the formulas above contain terms of order $\beta^0$, $\beta^1$, $\beta^2$ and $\beta^3$.

# 4 Amplitudes in a given $J^P$

We now fix the coordinate system in the overall center-of-mass system for $\bar{q}+q \to \gamma_1+\gamma_2$; the $z$-axis is fixed along $\vec{k}$, i.e. $\hat{z} = \hat{k}$. Then, we have

$$\begin{aligned}
\vec{p} &= E\beta(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\
\vec{k} &= E(0,0,1) \\
\varepsilon_1^*(\pm) = \varepsilon_2^*(\pm) &= \mp\frac{1}{\sqrt{2}}(1, \pm i, 0)^* = \frac{1}{\sqrt{2}}(\mp 1, i, 0)
\end{aligned} \tag{40}$$

where the $z$-components for spin ($\lambda_1 = \pm 1$ and $\lambda_2 = \pm 1$ for $\gamma_1$ and $\gamma_2$) are denoted in parentheses above. So the amplitude may be written

$$\mathcal{M}_\Lambda^{\rm nr}(m_1 m_2) = \left(\frac{Q^2}{E^2}\right)\left[\frac{F_{S\Lambda}^{\rm nr}(m_1 m_2) + \beta\cos\theta\, F_{A\Lambda}^{\rm nr}(m_1 m_2)}{1-\beta^2\cos^2\theta}\right]\beta^L \tag{41}$$

and

$$\begin{aligned}
F_{S\Lambda}^{\rm nr}(m_1 m_2) &= \chi^\dagger(m_1)\,(i\,\sigma_2)\,F_{S\Lambda}^{\rm nr}\,\chi(m_2) \\
F_{A\Lambda}^{\rm nr}(m_1 m_2) &= \chi^\dagger(m_1)\,(i\,\sigma_2)\,F_{A\Lambda}^{\rm nr}\,\chi(m_2)
\end{aligned} \tag{42}$$

where $m_1 = \pm 1/2$ and $m_2 = \pm 1/2$ specify the spin states of $\bar{q}$ and $q$, respectively. $\Lambda$ is the net helicity of the two-photon system, so that $\Lambda = 0$ or 2. The factor $\beta^L$ in (41) takes care of the barrier effect of the orbital angular momentum $L$. It is clear that the helicity-two and -zero states of the two photons in the final state arise from taking the following combination of polarization vectors:

$$\begin{aligned}
&\{\vec{\varepsilon}_1^{\,*}(+),\ \vec{\varepsilon}_2^{\,*}(+)\} \quad \text{for} \quad \Lambda = 2 \\
&\{\vec{\varepsilon}_1^{\,*}(+),\ \vec{\varepsilon}_2^{\,*}(-)\} \quad \text{for} \quad \Lambda = 0
\end{aligned} \tag{43}$$

So the amplitudes $F_{S\Lambda}^{\rm nr}$ and $F_{A\Lambda}^{\rm nr}$ in (42) with the subscripts $\Lambda$ are in fact the same as those in (39)—but with the combination of the polarization vectors as indicated above. The helicity amplitudes are related to the canonical ones via[6]

$$H_\Lambda^{\mu_1\mu_2}(\Omega) = \sum_{m_1 m_2} D^s_{m_1\mu_1}(\phi,\theta,0)\, D^s_{m_2\,-\mu_2}(\phi,\theta,0)\,\mathcal{M}_\Lambda^{\rm nr}(m_2 m_1) \tag{44}$$

and

$$\begin{aligned}
H_{S\Lambda}^{\mu_1\mu_2}(\Omega) &= \sum_{m_1 m_2} D^s_{m_1\mu_1}(\phi,\theta,0)\, D^s_{m_2\,-\mu_2}(\phi,\theta,0)\, F_{S\Lambda}^{\rm nr}(m_2 m_1) \\
H_{A\Lambda}^{\mu_1\mu_2}(\Omega) &= \sum_{m_1 m_2} D^s_{m_1\mu_1}(\phi,\theta,0)\, D^s_{m_2\,-\mu_2}(\phi,\theta,0)\, F_{A\Lambda}^{\rm nr}(m_2 m_1)
\end{aligned} \tag{45}$$

where $s=1/2$. Here we have chosen indices such that the set $\{m_1,\mu_1\}$ refers to $q(\Omega)$ and $\{m_2,\mu_2\}$ to $\bar{q}(\bar{\Omega})$.

In evaluating the formula above, it will turn out to be convenient to define several intermediate quantities. Define

$$\phi_\nu(m_1 m_2) = \chi^\dagger(m_1)\,(i\,\sigma_2)\,\sigma_\nu\,\chi(m_2) \tag{46}$$

where $\nu$ =0, 1, 2, and 3 and $\sigma_0$ is the $2\times 2$ identity matrix. We see that

$$\begin{aligned}\phi_\nu(\pm\pm) &= (0,\pm 1,+i,0)\\ \phi_\nu(\pm\mp) &= (\pm 1,0,0,-1)\end{aligned} \tag{47}$$

In analogy to (44), we may define

$$h_\nu^{\mu_1\mu_2}(\Omega) = \sum_{m_1 m_2} D^s_{m_1\mu_1}(\phi,\theta,0)\,D^s_{m_2\,-\mu_2}(\phi,\theta,0)\,\phi_\nu(m_2 m_1) \tag{48}$$

We see that

$$\begin{aligned}
h_\nu^{(++)}(\Omega) - h_\nu^{(--)}(\Omega) &= \phi_\nu(-+) - \phi_\nu(+-)\\
h_\nu^{(++)}(\Omega) + h_\nu^{(--)}(\Omega) &= -\sin\theta\left[\mathrm{e}^{-i\,\phi}\,\phi_\nu(++) - \mathrm{e}^{+i\,\phi}\,\phi_\nu(--)\right]\\
&\qquad + \cos\theta\left[\phi_\nu(+-)+\phi_\nu(-+)\right]\\
h_\nu^{(\pm\mp)}(\Omega) &= \frac{1}{2}\,\left[\mathrm{e}^{-i\,\phi}\,\phi_\nu(++) + \mathrm{e}^{+i\,\phi}\,\phi_\nu(--)\right]\\
&\quad \pm\frac{1}{2}\,\cos\theta\,\left[\mathrm{e}^{-i\,\phi}\,\phi_\nu(++) - \mathrm{e}^{+i\,\phi}\,\phi_\nu(--)\right]\\
&\qquad \pm\frac{1}{2}\,\sin\theta\,\left[\phi_\nu(+-)+\phi_\nu(-+)\right]
\end{aligned} \tag{49}$$

From these, we find [see Appendix A]

$$\begin{aligned}
h_\nu^{(++)}(\Omega) - h_\nu^{(--)}(\Omega) &= -2(1,0,0,0)\\
h_\nu^{(++)}(\Omega) + h_\nu^{(--)}(\Omega) &= -2(0,\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)\\
h_\nu^{(\pm\mp)}(\Omega) &= \Big\{0,\frac{1}{2}\left[\mathrm{e}^{-i\,\phi}\,(1\pm\cos\theta) - \mathrm{e}^{+i\,\phi}\,(1\mp\cos\theta)\right],\\
&\qquad \frac{i}{2}\left[\mathrm{e}^{-i\,\phi}\,(1\pm\cos\theta) + \mathrm{e}^{+i\,\phi}\,(1\mp\cos\theta)\right],\mp\sin\theta\Big\}\\
&= (0,\pm\cos\theta\cos\phi - i\,\sin\phi,\pm\cos\theta\sin\phi + i\,\cos\phi,\mp\sin\theta)
\end{aligned} \tag{50}$$

Noting that

$$\begin{aligned}\vec{p}\cdot\vec{\varepsilon}^{\,*}_{1,2}(\pm) &= \mp\frac{1}{\sqrt{2}}\,E\,\beta\,\sin\theta\,\mathrm{e}^{\mp i\,\phi}\\ \vec{p}\cdot\vec{k} &= E^2\beta\,\cos\theta\end{aligned} \tag{51}$$

and

$$
\begin{aligned}
&\left[\vec{h}^{(++)}(\Omega) - \vec{h}^{(--)}(\Omega)\right] \cdot \vec{\varepsilon}^{\,*}_{1,2}(\pm) = 0 \\
&\left[\vec{h}^{(++)}(\Omega) - \vec{h}^{(--)}(\Omega)\right] \cdot \vec{p} = 0, \quad \left[\vec{h}^{(++)}(\Omega) - \vec{h}^{(--)}(\Omega)\right] \cdot \vec{k} = 0 \\
&\left[\vec{h}^{(++)}(\Omega) + \vec{h}^{(--)}(\Omega)\right] \cdot \vec{\varepsilon}^{\,*}_{1,2}(\pm) = \pm\sqrt{2}\,\sin\theta\,\mathrm{e}^{\mp i\,\phi} \\
&\left[\vec{h}^{(++)}(\Omega) + \vec{h}^{(--)}(\Omega)\right] \cdot \vec{p} = -2\,E\,\beta, \quad \left[\vec{h}^{(++)}(\Omega) + \vec{h}^{(--)}(\Omega)\right] \cdot \vec{k} = -2\,E\,\cos\theta
\end{aligned}
\tag{52}
$$

and

$$
\begin{aligned}
\vec{h}^{(\pm\mp)}(\Omega) \cdot \vec{\varepsilon}^{\,*}_{1,2}(+) &= -\frac{1}{\sqrt{2}}\,(1 \pm \cos\theta)\,\mathrm{e}^{-i\,\phi} \\
\vec{h}^{(\pm\mp)}(\Omega) \cdot \vec{\varepsilon}^{\,*}_{1,2}(-) &= -\frac{1}{\sqrt{2}}\,(1 \mp \cos\theta)\,\mathrm{e}^{+i\,\phi} \\
\vec{h}^{(\pm\mp)}(\Omega) \cdot \vec{p} = 0, \quad \vec{h}^{(\pm\mp)}(\Omega) \cdot \vec{k} &= \mp E\,\sin\theta
\end{aligned}
\tag{53}
$$

we obtain [see (45) and (39)]

$$
\begin{aligned}
&H^{(++)}_{S2}(\Omega) - H^{(--)}_{S2}(\Omega) = 0, \quad H^{(++)}_{A2}(\Omega) - H^{(--)}_{A2}(\Omega) = 0 \\
&H^{(++)}_{S2}(\Omega) + H^{(--)}_{S2}(\Omega) = 2\,E\,\beta\sin^2\theta\,\mathrm{e}^{-2\,i\,\phi}, \quad H^{(++)}_{A2}(\Omega) + H^{(--)}_{A2}(\Omega) = 0 \\
&H^{(++)}_{S0}(\Omega) - H^{(--)}_{S0}(\Omega) = 2\,E, \quad H^{(++)}_{A0}(\Omega) - H^{(--)}_{A0}(\Omega) = 0 \\
&H^{(++)}_{S0}(\Omega) + H^{(--)}_{S0}(\Omega) = -2\,E\,\beta\sin^2\theta, \quad H^{(++)}_{A0}(\Omega) + H^{(--)}_{A0}(\Omega) = -2\,E\,\cos\theta
\end{aligned}
\tag{54}
$$

and

$$
\begin{aligned}
&H^{(\pm\mp)}_{S2}(\Omega) = -E\,\beta\,\left(1 + \frac{\beta^2}{2}\right)\,\sin\theta\,(1 \pm \cos\theta)\,\mathrm{e}^{-2\,i\,\phi}, \quad H^{(\pm\mp)}_{A2}(\Omega) = 0 \\
&H^{(\pm\mp)}_{S0}(\Omega) = \pm E\,\beta\,\left(1 + \frac{\beta^2}{2}\right)\,\sin\theta\,\cos\theta, \quad H^{(\pm\mp)}_{A0}(\Omega) = \mp E\,\left(1 + \frac{\beta^2}{2}\right)\,\sin\theta
\end{aligned}
\tag{55}
$$

Finally, we find

$$
\begin{aligned}
\int \mathrm{d}\phi\,\mathrm{e}^{2\,i\,\phi}\,\left[H^{(++)}_{2}(\Omega) - H^{(--)}_{2}(\Omega)\right] &= 0 \\
\int \mathrm{d}\phi\,\mathrm{e}^{2\,i\,\phi}\,\left[H^{(++)}_{2}(\Omega) + H^{(--)}_{2}(\Omega)\right] &= 2\Delta_L(\beta\cos\theta)\,\beta\,\sin^2\theta \\
\int \mathrm{d}\phi\,\left[H^{(++)}_{0}(\Omega) - H^{(--)}_{0}(\Omega)\right] &= 2\Delta_L(\beta\cos\theta) \\
\int \mathrm{d}\phi\,\left[H^{(++)}_{0}(\Omega) + H^{(--)}_{0}(\Omega)\right] &= -2\Delta_L(\beta\cos\theta)\,\beta \\
\int \mathrm{d}\phi\,\mathrm{e}^{2\,i\,\phi}\,H^{(\pm\mp)}_{2}(\Omega) = -\Delta_L(\beta\cos\theta)\,\beta\,&\left(1 + \frac{\beta^2}{2}\right)\,\sin\theta\,(1 \pm \cos\theta) \\
\int \mathrm{d}\phi\,H^{(\pm\mp)}_{0}(\Omega) &= 0
\end{aligned}
\tag{56}
$$

where

$$\Delta_L(\beta\cos\theta) = \frac{B_L(\beta)}{1-\beta^2\cos^2\theta}, \qquad B_L(\beta) = 2\pi\, N_L \left(\frac{Q^2}{E}\right) \beta^L \tag{57}$$

The amplitudes take on the form, from (14),

$$\begin{aligned}
\mathcal{M}_0^{\rm nr}(\,^1L_L) &= \sqrt{2}\int \mathrm{d}\cos\theta\,\Delta_L(\beta\cos\theta)\, d^L_{00}(\theta), \qquad \mathcal{M}_2^{\rm nr}(\,^1L_L) = 0 \\
\mathcal{M}_0^{\rm nr}(\,^3L_J) &= -\sqrt{2}\,(10L0|J0)\int \mathrm{d}\cos\theta\,\Delta_L(\beta\cos\theta)\,\beta\, d^J_{00}(\theta) = 0, \quad J = \text{odd} \\
\mathcal{M}_2^{\rm nr}(\,^3L_J) &= \sqrt{2}\,(10L0|J0)\int \mathrm{d}\cos\theta\,\Delta_L(\beta\cos\theta)\,\beta\, d^J_{20}(\theta)\,\sin^2\theta \\
&\quad -(11L0|J1)\int \mathrm{d}\cos\theta\,\Delta_L(\beta\cos\theta)\,\beta\left(1+\frac{\beta^2}{2}\right) \\
&\qquad \times\left[d^J_{21}(\theta)\,\sin\theta\,(1+\cos\theta) + (-)^J\, d^J_{2\,-1}(\theta)\,\sin\theta\,(1-\cos\theta)\right]
\end{aligned} \tag{58}$$

Note that the latter equation above can be transformed into [see Appendix B]

$$\begin{aligned}
\mathcal{M}_2^{\rm nr}(\,^3L_J) = 2\int \mathrm{d}\cos\theta\,\Delta_L(\beta\cos\theta)\,\beta\Big\{&\frac{2}{\sqrt{3}}\,(10L0|J0)\,d^2_{20}(\theta)\,d^J_{20}(\theta) \\
&+\left(1+\frac{\beta^2}{2}\right)(11L0|J1)\left[d^2_{21}(\theta)\,d^J_{21}(\theta) + (-)^J\, d^2_{2\,-1}(\theta)\,d^J_{2\,-1}(\theta)\right]\Big\}
\end{aligned} \tag{59}$$

We use the formula

$$d^\sigma_{\mu m}(\theta)\,d^J_{\mu m}(\theta) = \sum_\ell \left(\frac{2\ell+1}{2J+1}\right)(\sigma\mu\,\ell 0|J\mu)(\sigma m\,\ell 0|Jm)\,d^\ell_{00}(\theta) \tag{60}$$

where $\sigma = \mu = 2$ and $m = 0$, 1 or $-1$. We see that $|J-\sigma| \le \ell \le J+\sigma$. The equation (59) can be written

$$\begin{aligned}
\mathcal{M}_2^{\rm nr}(\,^3L_J) = 2\int \mathrm{d}\cos\theta\,\Delta_L(\beta\cos\theta)\,\beta\,&\sum_\ell\left(\frac{2\ell+1}{2J+1}\right)(22\ell 0|J2) \\
&\times\Big\{\frac{2}{\sqrt{3}}\,(10L0|J0)\,(20\ell 0|J0) \\
&\qquad +\left[1+(-)^\ell\right]\left(1+\frac{\beta^2}{2}\right)(11L0|J1)\,(21\ell 0|J1)\Big\}\,d^\ell_{00}(\theta)
\end{aligned} \tag{61}$$

We define

$$f_\ell(\beta) = \frac{1}{2}\int \mathrm{d}\cos\theta\,\frac{d^\ell_{00}(\theta)}{1-\beta^2\cos^2\theta} = 0, \quad \ell = \text{odd} \tag{62}$$

The amplitudes, integrated over $\cos\theta$, are

$$\begin{aligned}
\mathcal{M}_0^{\rm nr}(\,^1L_L) &= 2\sqrt{2}\,B_L(\beta)\,f_L(\beta), \qquad \mathcal{M}_2^{\rm nr}(\,^1L_L) = 0 \\
\mathcal{M}_0^{\rm nr}(\,^3L_J) &= -2\sqrt{2}\,(10L0|J0)\,B_L(\beta)\,\beta\,f_J(\beta) = 0, \quad J = \text{odd} \\
\mathcal{M}_2^{\rm nr}(\,^3L_J) &= 8B_L(\beta\cos\theta)\,\beta \sum_{\ell=\text{even}} \left(\frac{2\ell+1}{2J+1}\right)(22\ell 0|J2) \\
&\quad \times \left\{\frac{1}{\sqrt{3}}\,(10L0|J0)\,(20\ell 0|J0)\right. \\
&\qquad \left. + \left(1+\frac{\beta^2}{2}\right)(11L0|J1)\,(21\ell 0|J1)\right\} f_\ell(\beta)
\end{aligned} \tag{63}$$

See Appendix B for a tabulation of a few $f_\ell(\beta)$'s.

The reduced widths are defined via

$$\mathring{\Gamma}_\Lambda^{\rm nr}(\,^{2S+1}L_J) = \frac{1}{2J+1}\left|\mathring{\mathcal{M}}_\Lambda^{\rm nr}(\,^{2S+1}L_J)\right|^2_{\beta\to 0} \tag{64}$$

where the reduced amplitudes $\mathring{\mathcal{M}}_\Lambda^{\rm nr}(\,^{2S+1}L_J)$ are to be defined later.

# 5 Applications

We will work out the amplitudes $\mathcal{M}_\Lambda^{\rm nr}(\,^{2S+1}L_J)$ for $S$-, $P$-, $D$- and $F$-wave quarkonia. We shall use the formula (63), using the expansion for $f_\ell(\beta)$ as given in Appendix B. It may be worth noting that we need only the first two $f_\ell(\beta)$'s, i.e. $f_0(\beta)$ and $f_2(\beta)$. expanded up to $\beta^4$ because of the cancellation of the terms with lower-order powers of $\beta$. $f_4(\beta)$ appears only bcause it contributes to the correction terms.

For $L = 0$ we have

$$\mathcal{M}_0^{\rm nr}(\,^1S_0) = a\,f_0(\beta), \qquad \mathcal{M}_2^{\rm nr}(\,^1S_0) = 0 \tag{65}$$

where

$$a = 2\sqrt{2\pi}\left(\frac{Q^2}{E}\right) \tag{66}$$

and [see (62)]

$$\begin{aligned}
f_0(\beta) &= \frac{1}{2}\int \mathrm{d}\cos\theta\,\frac{1}{1-\beta^2\cos^2\theta} \\
&= \frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) \\
&= 1 + \frac{\beta^2}{3} + \frac{\beta^4}{5} + \frac{\beta^6}{7} + \frac{\beta^8}{9} + \cdots
\end{aligned} \tag{67}$$

So $f_0(\beta) \to 1$ as $\beta \to 0$. Note that $a$ is a constant since $E = M/2$ where $M$ is the quarkonium mass. For $L = 1$ we see that

$$\begin{aligned}
&\mathcal{M}_0^{\rm nr}({}^3P_0) = a\,\beta^2\, f_0(\beta), \quad \mathcal{M}_2^{\rm nr}({}^3P_0) = 0 \\
&\mathcal{M}_0^{\rm nr}({}^3P_1) = 0, \qquad \mathcal{M}_2^{\rm nr}({}^3P_1) = 0 \\
&\mathcal{M}_0^{\rm nr}({}^3P_2) = -\sqrt{2}\,a\,\beta^2\, f_2(\beta) \\
&\mathcal{M}_2^{\rm nr}({}^3P_2) = \frac{2}{\sqrt{3}}\,a\,\beta^2 \left[(1+\frac{3}{10}\,\beta^2)\, f_0(\beta) - (1+\frac{3}{14}\,\beta^2) f_2(\beta) - \frac{3}{35}\, f_4(\beta)\right]
\end{aligned} \tag{68}$$

It is informative to see what happens to the amplitudes above if $f_0(\beta)$ is expanded to order $\beta^2$:

$$\begin{aligned}
\mathcal{M}_0^{\rm nr}({}^3P_0) &\simeq a\,\beta^2 \left(1 + \frac{\beta^2}{3}\right) \\
\mathcal{M}_0^{\rm nr}({}^3P_2) &\simeq -\frac{2\sqrt{2}}{15}\,a\,\beta^4 \\
\mathcal{M}_2^{\rm nr}({}^3P_2) &\simeq \frac{2}{\sqrt{3}}\,a\,\beta^2 \left(1 + \frac{\beta^2}{2}\right)
\end{aligned} \tag{69}$$

Retaining terms to $\beta^2$ only, we see that

$$\begin{aligned}
\mathcal{M}_0^{\rm nr}({}^3P_0) \,:\, \mathcal{M}_2^{\rm nr}({}^3P_0) \,:&\mathcal{M}_0^{\rm nr}({}^3P_1) \,:\, \mathcal{M}_2^{\rm nr}({}^3P_1) \,:\, \mathcal{M}_0^{\rm nr}({}^3P_2) \,:\, \mathcal{M}_2^{\rm nr}({}^3P_2) \\
&\simeq 1 \,:\, 0 : \,0 : \,0 : 0 \,:\, \frac{2}{\sqrt{3}}
\end{aligned} \tag{70}$$

We thus conclude that the ${}^3P_2$ quarkonia coupling to two photons occur predominantly in helicity $\pm 2$, a well known result. The reduced widths have the following relationship:

$$\begin{aligned}
\mathring{\Gamma}_0^{\rm nr}({}^3P_0) \,:\, \mathring{\Gamma}_2^{\rm nr}({}^3P_0) \,:&\mathring{\Gamma}_0^{\rm nr}({}^3P_1) \,:\, \mathring{\Gamma}_2^{\rm nr}({}^3P_1) \,:\, \mathring{\Gamma}_0^{\rm nr}({}^3P_2) \,:\, \mathring{\Gamma}_2^{\rm nr}({}^3P_2) \\
&\simeq \frac{15}{4} \,:\, 0 \,:\, 0 \,:\, 0 \,:\, 0 \,:\, 1
\end{aligned} \tag{71}$$

Again, this is a well-known nonrelativistic result. For $L = 2$ we have

$$\mathcal{M}_0^{\rm nr}({}^1D_2) = \sqrt{5}\,a\,\beta^2\, f_2(\beta), \qquad \mathcal{M}_2^{\rm nr}({}^1D_2) = 0 \tag{72}$$

so that

$$\mathcal{M}_0^{\rm nr}({}^1D_2) \simeq \frac{2}{3\sqrt{5}}\,a\,\beta^4\,(1 + \frac{12}{7}\,\beta^2) \tag{73}$$

Finally, for $F$-wave quarkonia we need to retain only the $\beta^6$ terms in the expansion in $\beta$ as $\beta \to 0$ but carry along the $\beta^8$ terms to show the amount of required correction. For $J = 2$ we have

$$\begin{aligned}
&\mathcal{M}_0^{\rm nr}({}^3F_2) = \sqrt{3}\,a\,\beta^4\, f_2(\beta) \\
&\mathcal{M}_2^{\rm nr}({}^3F_2) = \sqrt{2}\,a\,\beta^4 \left[\frac{1}{5}\,\beta^2\, f_0(\beta) + \frac{2}{7}\,(1-\frac{1}{2}\,\beta^2)\, f_2(\beta) - \frac{2}{7}\,(1+\frac{1}{5}\,\beta^2) f_4(\beta)\right]
\end{aligned} \tag{74}$$

and for $J = 3$ we find

$$\begin{aligned}
&\mathcal{M}_0^{\rm nr}(\,^3F_3) = 0 \\
&\mathcal{M}_2^{\rm nr}(\,^3F_3) = \sqrt{\frac{10}{7}}\, a\, \beta^4\, (1 + \frac{1}{2}\,\beta^2)\ [f_2(\beta) - f_4(\beta)]
\end{aligned} \tag{75}$$

For $J = 4$ we get

$$\begin{aligned}
&\mathcal{M}_0^{\rm nr}(\,^3F_4) = -2a\, \beta^4\, f_4(\beta) \\
&\mathcal{M}_2^{\rm nr}(\,^3F_4) = \frac{3\sqrt{10}}{7}\, a\, \beta^4\, \left[(1 + \frac{5}{18}\,\beta^2)\, f_2(\beta) - (1 + \frac{3}{22}\,\beta^2) f_4(\beta) - \frac{14}{99}\, f_6(\beta)\right]
\end{aligned} \tag{76}$$

We see that, dropping terms of order $\beta^{10}$ or higher,

$$\begin{aligned}
&\mathcal{M}_0^{\rm nr}(\,^3F_2) \simeq \frac{2}{5\sqrt{3}}\, a\, \beta^6\, (1 + \frac{12}{7}\,\beta^2) \\
&\mathcal{M}_2^{\rm nr}(\,^3F_2) \simeq \frac{5\sqrt{2}}{21}\, a\, \beta^6\, (1 + \frac{89}{525}\,\beta^2) \\
&\mathcal{M}_0^{\rm nr}(\,^3F_3) = 0, \quad \mathcal{M}_2^{\rm nr}(\,^3F_3) \simeq \frac{2}{3}\sqrt{\frac{2}{35}}\, a\, \beta^6\, (1 + \frac{13}{42}\,\beta^2) \\
&\mathcal{M}_0^{\rm nr}(\,^3F_4) \simeq -\frac{16}{315}\, a\, \beta^8, \quad \mathcal{M}_2^{\rm nr}(\,^3F_4) \simeq \frac{2}{7}\sqrt{\frac{2}{5}}\, a\, \beta^6\, (1 + \frac{227}{126}\,\beta^2)
\end{aligned} \tag{77}$$

The relative ratios are, retaining $\beta^6$ terms only,

$$\begin{aligned}
&\mathcal{M}_0^{\rm nr}(\,^3F_2) : \mathcal{M}_2^{\rm nr}(\,^3F_2) : \mathcal{M}_0^{\rm nr}(\,^3F_3) : \mathcal{M}_2^{\rm nr}(\,^3F_3) : \mathcal{M}_0^{\rm nr}(\,^3F_4) : \mathcal{M}_2^{\rm nr}(\,^3F_4) \\
&\qquad \simeq \sqrt{\frac{21}{10}} : \frac{5}{2}\sqrt{\frac{5}{7}} : 0 : 1 : 0 : \frac{3}{\sqrt{7}}
\end{aligned} \tag{78}$$

and the reduced widths are

$$\begin{aligned}
&\mathring{\Gamma}_0^{\rm nr}(\,^3F_2) : \mathring{\Gamma}_2^{\rm nr}(\,^3F_2) : \mathring{\Gamma}_0^{\rm nr}(\,^3F_3) : \mathring{\Gamma}_2^{\rm nr}(\,^3F_3) : \mathring{\Gamma}_0^{\rm nr}(\,^3F_4) : \mathring{\Gamma}_2^{\rm nr}(\,^3F_4) \\
&\qquad \simeq \frac{3}{2}\left(\frac{7}{5}\right)^2 : \left(\frac{5}{2}\right)^2 : 0 : 1 : 0 : 1
\end{aligned} \tag{79}$$

It is seen that the $F$-wave quarkonium with helicity $\pm 2$ for the two photons dominate over those with helicity zero. In particular, the $^3F_4$ state with helicity zero does not couple to two photons in the nonrelativistic limit. It may be worth noting that, although the partial width for $^3F_2$ (helicity 0) is nonzero, those for $^3F_2$ (helicity $+2$ plus $-2$) are over four times the helicity-zero width.

# 6 Conclusions

We have carried out a calculation of the two-photon coupling of quarkonia ${}^{2S+1}L_J$ for an arbitrary $L$ and the results are applied to $L \leq 3$, within a model in which the lowest-order QED diagram for for the process $q + \bar{q} \to \gamma + \gamma$ has been used.

The derivation relies on the use of Pauli spinors, since the process involves summation of quark-spin states within an amplitude. It should be emphasized that the spin projection operators which involve integration over angles *require* both $\theta$ and $\phi$; for instance, $h_\nu^{(\pm\mp)}(\Omega)$ [see (50)] do appear with different $\phi$ dependence. It is clear, therefore, that we are not dealing with the problem of evaluating the standard two-body scattering process $q + \bar{q} \to \gamma + \gamma$.

The nonrelativistic amplitudes in our model, after integration over $\phi$, may be expressed succinctly as [see (58) and (59)]

$$\mathcal{M}_\Lambda^{\mathrm{nr}}({}^{2S+1}L_J \,|\, \theta) = \Delta_L(\beta\cos\theta)\,\beta^S \sum_\mu g_\mu^\Lambda({}^{2S+1}L_J)\, d_{\Lambda\mu}^\Lambda(\theta)\, d_{\Lambda\mu}^J(\theta) \tag{80}$$

where $\Delta_L(\beta\cos\theta)$ is the propagator and $g_\mu^\Lambda$'s are the constants. Here $\theta$ is the scattering angle for the process $q + \bar{q} \to \gamma + \gamma$; $\Lambda$ (0 or 2) is the net helicity along the two-photon axis; and $\mu$ (0 or $\pm 1$) is the net helicity along the $q\bar{q}$ axis. For the record—and to summarize—we specify completely the amplitudes in our model:

$$\begin{aligned}
&\Delta_L(\beta\cos\theta) = \frac{B_L(\beta)}{1 - \beta^2\cos^2\theta}, \qquad B_L(\beta) = \sqrt{\pi\,(2L+1)}\left(\frac{Q^2}{E}\right)\beta^L \\
&g_0^0({}^1L_L) = \sqrt{2}, \qquad g_\mu^2({}^1L_L) = 0 \\
&g_0^0({}^3L_J) = -\sqrt{2}\,(10\,L0|J0) = 0, \qquad J = \text{odd} \\
&g_0^2({}^3L_J) = \frac{4}{\sqrt{3}}\,(10\,L0|J0) = 0, \qquad J = \text{odd} \\
&g_1^2({}^3L_J) = (2+\beta^2)\,(11\,L0|J1) \\
&g_{-1}^2({}^3L_J) = (2+\beta^2)\,(1\,{-1}L0|J\,{-1})
\end{aligned} \tag{81}$$

where $L$ = even for $S = 0$ and $L$ = odd for $S = 1$. If $\beta$ is sufficiently small, then the propagator becomes independent of $\theta$. In this limit, the orthogonality of the $d$-functions demands that $J = \Lambda$. So $\mathcal{M}_0^{\mathrm{nr}}({}^{2S+1}L_J) \to 0$ unless $J = 0$. Likewise, $\mathcal{M}_2^{\mathrm{nr}}({}^{2S+1}L_J) \to 0$ unless $J = 2$. This shows that spin-0 and spin-2 states should be dominant in general. Merely by counting the powers of $\beta$, we deduce that the partial width for $J^{PC} = 0^{-+}\,({}^1S_0)$

should be larger than those for $J^{PC} = 0^{++}\,(^3P_0)$ and $J^{PC} = 2^{++}\,(^3P_2)$, while the partial width for $J^{PC} = 0^{++}$ is expected to be nearly four times bigger than that for $J^{PC} = 2^{++}$. In addition, the tensor states are produced with the two-photon helicities $\pm 2$.

It has been shown that the amplitudes can be made to be proportional to a sum of the Legendre polynomials $P_\ell(\cos\theta) = d^\ell_{00}(\theta)$, where $\ell$ ranges from $|J-\Lambda|$ to $J+\Lambda$ [see (61)]. The nonrelativistic amplitudes, after integration over the angle $\theta$ and in the limit $\beta \to 0$, show a $\beta^{2L}$ dependence, but this result emerges only after intricate cancellation of the lower-order $\beta$ terms in the derivation.

The reduced widths as given by Ackleh, Barnes and Close[4] are summarized in table II for $L \leq 3$.

Table II: Reduced Partial Widths
for $L \leq 3$ from Ref. [4]

| $J^{PC}$ | $^{2S+1}L_J$ | $\Lambda$ | $\mathring{\Gamma}_\Lambda(^{2S+1}L_J)$ |
|---|---|---|---|
| $0^{-+}$ | $^1S_0$ | 0 | 1 |
| $0^{-+}$ | $^1S_0$ | 2 | 0 |
| $0^{++}$ | $^3P_0$ | 0 | 9 |
| $0^{++}$ | $^3P_0$ | 2 | 0 |
| $2^{++}$ | $^3P_2$ | 0 | 0 |
| $2^{++}$ | $^3P_2$ | 2 | 12/5 |
| $2^{-+}$ | $^1D_2$ | 0 | 1 |
| $2^{-+}$ | $^1D_2$ | 2 | 0 |
| $2^{++}$ | $^3F_2$ | 0 | 49/15 |
| $2^{++}$ | $^3F_2$ | 2 | 128/18 |
| $3^{++}$ | $^3F_3$ | 0 | 0 |
| $3^{++}$ | $^3F_3$ | 2 | 10/9 |
| $4^{++}$ | $^3F_4$ | 0 | 0 |
| $4^{++}$ | $^3F_4$ | 2 | 10/9 |

It is seen that our results agree with these within each $L$-wave quarkonium family.

## Acknowledgment

I thank Ted Barnes for his very helpful comments. I am indebted Bob Hackenburg for his careful reading of an earlier version of this paper and his numerous suggestions.

## Appendix A

The Dirac equation for an electron in an electromagnetic field $A$ is

$$[\gamma^\mu (i\,\partial_\mu + e\,A) - m]\ \psi = 0 \tag{82}$$

Its complex conjugate is

$$[-\gamma^{\mu\,*} (i\,\partial_\mu - e\,A) - m]\ \psi^* = 0 \tag{83}$$

Now the Dirac equation for a positron must take on the form

$$[\gamma^\mu (i\,\partial_\mu - e\,A) - m]\ \psi_C = 0 \tag{84}$$

If $\psi$ is the wave function describing an electron, then $\psi_C$ is its analogue for a positron. We use the Dirac-Pauli representation of $\gamma^\mu$, i.e.

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \qquad \text{and} \qquad \gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{85}$$

and

$$\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu} \qquad \text{and} \qquad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{86}$$

Note that in this representation a *real* matrix $\mathcal{C} = i\,\gamma^2$ has the property such that

$$-\mathcal{C}\,\gamma^{\mu\,*} = \gamma^\mu\,\mathcal{C} \qquad \text{and} \qquad \mathcal{C}^2 = -1 \tag{87}$$

So we see that

$$\psi_C = \mathcal{C}\,\psi^* \tag{88}$$

We define a free-particle spinor [see (20)] in its usual form

$$u(\vec{p}) = \left(\frac{E+m}{2m}\right)^{1/2} \begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \chi \tag{89}$$

where $\psi = u(\vec{p}) \exp(-i\, p \cdot x)$. The Dirac equation for electrons in the momentum space is

$$(\not{p} - m) u(\vec{p}) = 0 \tag{90}$$

(88) implies that [again, see (20)]

$$v(\vec{p}) = \mathcal{C}\, u^*(\vec{p}) = \left(\frac{E+m}{2m}\right)^{1/2} \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix} (-i\,\sigma_2)\,\chi \tag{91}$$

where $\psi_C = v(\vec{p}) \exp(+i\, p \cdot x)$. The Dirac equation for positrons takes on the form

$$(\not{p} + m) v(\vec{p}) = 0 \tag{92}$$

which is obtained by applying $\mathcal{C}$ from the left on the complex conjugate of (90) or, equivalently, by applying $\mathcal{C}$ from the left on (83).

Let us recall the angular-momentum operator $\vec{J}$ in its *standard representation.* The Hermitian operators satisfy

$$[J_i, J_k] = i\,\varepsilon_{ijk}\, J_k \tag{93}$$

The ket states $|jm\rangle$ in the standard representation satisfy

$$\begin{aligned} (\vec{J})^2\, |jm\rangle &= j(j+1)\, |jm\rangle \\ J_3\, |jm\rangle &= m\, |jm\rangle \\ J_\pm\, |jm\rangle &= \sqrt{(j \mp m)(j \pm m + 1}\, |jm\rangle \end{aligned} \tag{94}$$

where $J_\pm = J_1 \pm i\, J_2$. For spin-1/2 particles, these requirements are realized in the Pauli spinor basis via

$$\vec{J} = \frac{1}{2}\vec{\sigma}, \quad \left|\frac{1}{2}\ {+\frac{1}{2}}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi(+\tfrac{1}{2}) = \chi_+, \quad \left|\frac{1}{2}\ {-\frac{1}{2}}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \chi(-\tfrac{1}{2}) = \chi_- \tag{95}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{96}$$

Note that the factor $(-i\,\sigma_2)$ of (91) is simply a rotation by $+\pi$ around the $y$-axis, as seen from the relationship

$$R_2(\pm\pi) = \exp[\mp i\,\pi\,J_2] = \exp[\mp i\,\left(\frac{\pi}{2}\right)\,\sigma_2] = \cos\left(\frac{\pi}{2}\right) \mp i\,\sin\left(\frac{\pi}{2}\right)\,\sigma_2 = \mp i\,\sigma_2 \tag{97}$$

It is helpful to write down the transformation properties (94) for $\chi$'s. From (94), we see that

$$(\vec{\sigma}\,)^2\,\chi_\pm = 3\,\chi_\pm, \qquad \sigma_3\,\chi_\pm = \pm\,\chi_\pm, \qquad \sigma_\pm\,\chi_\pm = 0, \qquad \sigma_\pm\,\chi_\mp = 2\,\chi_\pm, \tag{98}$$

where $\sigma_\pm = \sigma_1 \pm i\,\sigma_2$. This can be independently verified, of course, starting with the definitions (95) and (96). Let $\phi$ be the Pauli spinors for the positrons. From (91), we see that

$$\phi_\mp = (-i\,\sigma_2)\,\chi_\pm, \quad \phi_- = \phi(-\frac{1}{2}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \quad \phi_+ = \phi(+\frac{1}{2}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{99}$$

and they obey

$$(\vec{\sigma}\,)^2\,\phi_\pm = 3\,\phi_\pm, \qquad \sigma_3\,\phi_\pm = \pm\,\phi_\pm, \qquad \sigma_\pm\,\phi_\pm = 0, \qquad \sigma_\pm\,\phi_\mp = -2\,\phi_\mp, \tag{100}$$

So $\phi$'s do not transform like the states in the standard representation. It is apparent, therefore, that $\phi$'s cannot be treated on an equal footing with $\chi$'s. Consider a problem at hand—the initial state consists of a $q$ and a $\bar{q}$—we could construct a state of a given intrinsic spin $S$ from $\chi^\dagger$ and $\chi$ by using the standard Clebsch-Gordan coefficients but *not* from $\phi^\dagger$ and $\chi$.

We now turn to a discussion of the helicity states in which the $z$ axis is along $\Omega(\theta,\phi)$. We use the convention in which a rotation (active) by $\Omega(\theta,\phi)$ acting on a canonical state $|jm\rangle$ at rest induces a transformation

$$R[\Omega(\theta,\phi)]\,|jm\rangle = \sum_{m'} |jm'\rangle\langle jm'|R[\Omega(\theta,\phi)]|jm\rangle = \sum_{m'} |jm'\rangle\, D^j_{m'm}(\phi,\theta,0) \tag{101}$$

We start with at-rest helicity states aligned with the 'space-fixed' coordinate system (in this case, the $z$-axis—originally chosen to be along the two-photon axis—coincides with the $q\bar{q}$ axis, with $q$ along $\hat{z}$ and $\bar{q}$ along $-\hat{z}$. The $q$ and $\bar{q}$ rest frames are reached by pure time-like Lorentz transformations from the $q\bar{q}$ rest frame.) Let $\mu_1$ be the helicity of $q$ and $\mu_2$ that of $\bar{q}$. The helicity states at rest may be written

$$\psi^s_{\mu_1}(\Omega_0) = \chi(\mu_1), \qquad \psi^s_{\mu_2}(\Omega_0) = \chi(-\mu_2) \tag{102}$$

where $s = 1/2$ and $\Omega_0(0,0)$. We note that our helicity states coincide with the canonical states $\chi$ defined in (95). Now apply a rotation $R$ on the states above; the helicities do not change by definition but $\Omega_0$ changes into $\Omega$, while the canonical $\chi$'s undergo transformations as prescribed in (101). So we obtain [see also (46) and (48)]

$$\begin{aligned} \psi^s_{\mu_1}(\Omega) &= \sum_{m_1} D^s_{m_1\,\mu_1}(\phi,\theta,0)\,\chi(m_1) \\ \bar{\psi}^s_{\mu_2}(\Omega) &= \sum_{m_2} D^s_{m_2\,-\mu_2}(\phi,\theta,0)\,\chi(m_2) \end{aligned} \tag{103}$$

where the spinors $\psi$ refer to $q$ in the direction $\Omega(\theta,\phi)$ and likewise $\bar{\psi}$ to $\bar{q}$, so the direction of $\bar{q}$ is along $-\vec{q}$. We see that

$$\psi^s_+(\Omega) = \bar{\psi}^s_-(\Omega) = \begin{pmatrix} v \\ u^* \end{pmatrix}, \qquad \psi^s_-(\Omega) = \bar{\psi}^s_+(\Omega) = \begin{pmatrix} -u \\ v^* \end{pmatrix} \tag{104}$$

where

$$u = \sin(\theta/2)\,\mathrm{e}^{-i\,(\phi/2)}, \qquad v = \cos(\theta/2)\,\mathrm{e}^{-i\,(\phi/2)} \tag{105}$$

So the quantity $h_\nu$ of (48) may be expressed

$$h^{\mu_1\mu_2}_\nu(\Omega) = [\bar{\psi}^s_{\mu_2}(\Omega)]^T\,(i\,\sigma_2)\,\sigma_\nu\;\psi^s_{\mu_1}(\Omega) \tag{106}$$

Here the superscript $T$ stands for the 'transpose'—we do *not* use the Hermitian conjugate because both $q$ and $\bar{q}$ represent the initial state; this is in contrast to the Feynmann rules governing the treatment of particles and antiparticles. We find

$$\begin{aligned} h^{(++)}_\nu(\Omega) &= (-u \quad v^*)\;(i\,\sigma_2)\,\sigma_\nu \begin{pmatrix} v \\ u^* \end{pmatrix}, \quad h^{(--)}_\nu(\Omega) = (v \quad u^*)\;(i\,\sigma_2)\,\sigma_\nu \begin{pmatrix} -u \\ v^* \end{pmatrix} \\ h^{(+-)}_\nu(\Omega) &= (v \quad u^*)\;(i\,\sigma_2)\,\sigma_\nu \begin{pmatrix} v \\ u^* \end{pmatrix}, \quad h^{(-+)}_\nu(\Omega) = (-u \quad v^*)\;(i\,\sigma_2)\,\sigma_\nu \begin{pmatrix} -u \\ v^* \end{pmatrix} \end{aligned} \tag{107}$$

so that

$$\begin{aligned} h^{(++)}_0(\Omega) &= -1, \quad h^{(--)}_0(\Omega) = +1 \\ h^{(++)}_1(\Omega) &= -\sin\theta\cos\phi, \quad h^{(--)}_1(\Omega) = -\sin\theta\cos\phi \\ h^{(++)}_2(\Omega) &= -\sin\theta\sin\phi, \quad h^{(--)}_2(\Omega) = -\sin\theta\sin\phi \\ h^{(++)}_3(\Omega) &= -\cos\theta, \quad h^{(--)}_3(\Omega) = -\cos\theta \end{aligned} \tag{108}$$

and

$$
\begin{aligned}
h_0^{(\pm\mp)}(\Omega) &= 0 \\
h_1^{(\pm\mp)}(\Omega) &= \frac{1}{2}\left[\mathrm{e}^{-i\,\phi}\,(1 \pm \cos\theta) - \mathrm{e}^{+i\,\phi}\,(1 \mp \cos\theta)\right] \\
h_2^{(\pm\mp)}(\Omega) &= \frac{i}{2}\left[\mathrm{e}^{-i\,\phi}\,(1 \pm \cos\theta) + \mathrm{e}^{+i\,\phi}\,(1 \mp \cos\theta)\right] \\
h_3^{(\pm\mp)}(\Omega) &= \mp \sin\theta
\end{aligned}
\tag{109}
$$

We see that (108) and (109) are identical to (50).

# Appendix B

A collection of relevant $d^J_{m'm}(\theta)$ functions are given here. We start with, for $J \leq 2$,

$$
\begin{aligned}
&d^0_{00}(\theta) = 1 \\
&d^1_{00}(\theta) = \cos\theta, \quad d^1_{10}(\theta) = -\frac{1}{\sqrt{2}}\,\sin\theta \\
&d^2_{00}(\theta) = \frac{1}{2}\,(3\cos^2 - 1), \quad d^2_{10}(\theta) = -\sqrt{\frac{3}{2}}\,\sin\theta\cos\theta, \quad d^2_{20}(\theta) = \frac{1}{2}\sqrt{\frac{3}{2}}\,\sin^2\theta \\
&d^2_{21}(\theta) = -\frac{1}{2}\sin\theta\,(1+\cos\theta), \quad d^2_{2\,-1}(\theta) = -\frac{1}{2}\sin\theta\,(1-\cos\theta)
\end{aligned}
\tag{110}
$$

For $J = 3$ and $J = 4$, we have

$$
\begin{aligned}
d^3_{00}(\theta) &= \frac{1}{2}\,(5\cos^3\theta - 3\cos\theta), \quad d^3_{10}(\theta) = -\frac{\sqrt{3}}{4}\,\sin\theta\,(5\cos^2\theta - 1) \\
d^3_{20}(\theta) &= \frac{1}{2}\sqrt{\frac{15}{2}}\,\sin^2\theta\,\cos\theta, \quad d^3_{30}(\theta) = -\frac{\sqrt{5}}{4}\,\sin^3\theta \\
d^3_{21}(\theta) &= -\frac{1}{4}\sqrt{\frac{5}{2}}\sin\theta\,(3\cos\theta - 1)\,(1+\cos\theta) \\
d^3_{2\,-1}(\theta) &= -\frac{1}{4}\sqrt{\frac{5}{2}}\,\sin\theta\,(3\cos\theta + 1)\,(1-\cos\theta) \\
d^4_{00}(\theta) &= \frac{1}{8}\,(35\cos^4\theta - 30\cos^2 + 3), \quad d^4_{10}(\theta) = -\frac{\sqrt{5}}{4}\,\sin\theta\,\cos\theta\,(7\cos^2\theta - 3) \\
d^4_{20}(\theta) &= \frac{1}{4}\sqrt{\frac{5}{2}}\,\sin^2\theta\,(7\cos^2\theta - 1) \\
d^4_{21}(\theta) &= -\frac{\sqrt{2}}{8}\,\sin\theta\,(14\cos^2\theta - 7\cos\theta - 1)\,(1+\cos\theta) \\
d^4_{2\,-1}(\theta) &= -\frac{\sqrt{2}}{8}\,\sin\theta\,(14\cos^2\theta + 7\cos\theta - 1)\,(1-\cos\theta)
\end{aligned}
\tag{111}
$$

Finally, we need, for $J = 6$,

$$\begin{aligned} d^6_{00}(\theta) &= \frac{1}{16}\,(231\cos^6\theta - 315\cos^4\theta + 105\cos^2\theta - 5) \\ d^6_{20}(\theta) &= -\frac{\sqrt{105}}{32}\,\sin^2\theta\,(33\cos^4\theta - 18\cos^2\theta + 1) \end{aligned} \tag{112}$$

We can now evaluate $f_\ell(\beta)$'s for a few values of $\ell$ [see (62)]. They are

$$\begin{aligned} f_0(\beta) &= \frac{1}{2\beta}\,\ln\left(\frac{1+\beta}{1-\beta}\right) \\ f_2(\beta) &= \frac{1}{2\beta^2}\,\left\{3\,[f_0(\beta) - 1] - \beta^2\,f_0(\beta)\right\} \\ f_4(\beta) &= \frac{1}{24\beta^4}\,\left\{105\,[f_0(\beta) - 1] - 5\beta^2\,[18 f_0(\beta) - 11] + 9\beta^4\,f_0(\beta)\right\} \\ f_6(\beta) &= \frac{1}{80\beta^6}\,\left\{1155\,[f_0(\beta) - 1] - 35\beta^2\,[45 f_0(\beta) - 34]\right. \\ &\qquad\qquad \left. +21\beta^4\,[25 f_0(\beta) - 11] - 25\beta^6\,f_0(\beta)\right\} \end{aligned} \tag{113}$$

We expand $f_\ell(\beta)$ in powers of $\beta^{2n}$ for integer $n$ until the first nonzero term is encountered; we find that $n \in \{0, 6\}$. The results are listed below:

$$\begin{aligned} f_0(\beta) &\simeq \frac{1}{1}\,\beta^0 \\ f_2(\beta) &\simeq \frac{2}{15}\,\beta^2\,(1 + \frac{12}{7}\,\beta^2) = \frac{2}{3\cdot 5}\,\beta^2\,(1 + \frac{12}{7}\,\beta^2) \\ f_4(\beta) &\simeq \frac{8}{315}\,\beta^4 = \frac{8}{3^2\cdot 5\cdot 7}\,\beta^4 \\ f_6(\beta) &\simeq \frac{16}{3003}\,\beta^6 = \frac{16}{3\cdot 7\cdot 11\cdot 13}\,\beta^6 \end{aligned} \tag{114}$$

Note that $f_0(\beta)$ has already been given in (67), but we give it here again in a form consistent with the others.

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