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Positivity conditions on the spin density matrix: A simple parametrization*

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A simple parametrization of the spin density matrix ρ is given which ensures that ρ is a positive, Hermitian matrix which satisfies the parity symmetry conditions. There is a minimal number of parameters and they can vary independently; no complicated constraints on their allowed region of variation are required. A similar parametrization is given for the real part of ρ. Rank conditions, polarized targets, and angular distributions are also discussed.

I. INTRODUCTION

In the determination of the spin density matrix ρ of a particle, a resonance, or a more general system from experimental data, it is very useful to have a parametrization which ensures that all the constraints to which ρ is subject are satisfied.[1-6] Ideally, one would like a parametrization using the minimal number of parameters which automatically yields a Hermitian ρ satisfying the positivity conditions, parity conditions, and rank conditions, such as the Eberhard-Good theorem.[5] This parametrization should be useful both for the case where the system in question has a definite spin and for the case where it is a mixture of different spins.

The main purpose of this paper is to present a simple parametrization of ρ for a system produced in a parity-conserving process such that the parameters can vary *independently* over a preassigned range (say, between -1 and $+1$ or 0 and 2π) and always yield a ρ such that

(a) ρ is Hermitian, $\rho = \rho^\dagger$;
(b) ρ satisfies the symmetry imposed by parity conservation in the production process;
(c) the eigenvalues of ρ, which are real, are non-negative, and the matrix ρ is "positive";
(d) the rank r of ρ is in some cases smaller than the dimension of ρ.

Most commonly, ρ is parametrized so that (a) and (b) are satisfied and then (c) and (d) are imposed as constraints. Daboul[6] has given several simple parametrizations for which (a) and (c) are satisfied but then (b) is imposed as a constraint. Both procedures have the disadvantage that the ranges over which the parameters can vary are constrained in a complicated way.

The parametrization we shall present here is a generalization of that given by Daboul—he calls it the Cholesky decomposition of ρ—which guarantees that (a)–(d) are satisfied in a very simple way. It has the further practical advantage of utilizing parity conservation (b) to break up the matrix into matrices of smaller dimension. Furthermore, when the system has a definite spin, the number of parameters used is equal to the number of measurable quantities. For the case of mixtures of spins we generally have some extra parameters. This method works for joint density matrices as well.

We consider in this paper the density matrix ρ of a resonance, or of a mixture of resonances and background R, produced in a parity-conserving process $A+B \to R+C$. (The nature of C is not important for these considerations. For definiteness we assume it to be a system of definite mass, spin, and parity.) The key to our simplified parametrization is to introduce eigenstates of the reflection operator Π_y corresponding to reflections in the production plane. The idea of employing "parity eigenstates" has been around for some time.[7] The application of this idea to the density-matrix analysis has been carried out in the case of bosons by Ascoli.[8]

We start in Sec. II by defining a complete set of orthonormal eigenstates of the reflection operator. We then express ρ in terms of the reflection eigenstates and give the connection to the conventional basis. We review briefly the relation of ρ to angular distributions and discuss the measurability of the various elements of ρ. In Sec. III we discuss the positivity conditions and present our parametrization. Rank conditions require a simple restriction on this parametrization and are given in Sec. IV. Section V gives a brief account of how some of these results are modified when a polarized target is used. Section VI is devoted to a few important examples for both bosons and fermions as illustrations of this method. We give in the Appendix the angular distributions for a wide class of resonance decays and examine in some detail the constraints due to parity conservation.

II. PARITY CONSERVATION AND THE REFLECTION OPERATOR

It has become a common practice to use the spin density matrix ρ in the study of resonance production with the rest frame for the resonance chosen according to either the Gottfried-Jackson[9] (GJ) or the s-channel helicity convention. Both of these conventions, used for defining the spin quantization axis of R in the process $A+B\to R+C$, belong to a larger class of conventions in which the y axis is normal to the production plane and the z axis (i.e., the quantization axis) lies along some preassigned direction in the production plane. That direction is $\vec{p}_A$ in the GJ convention and $-\vec{p}_C$ in the s-channel helicity convention, where $\vec{p}_A$ and $\vec{p}_C$ denote the momenta of A and C in the rest frame of R.

The operator Π_y of reflections in the production plane is related to parity by a rotation through π about the y axis:

$$\Pi_y = Pe^{-i\pi J_y}. \tag{2.1}$$

The helicity states as defined by Jacob and Wick[10] satisfy

$$\Pi_y|\vec{p},a,\lambda\rangle = \eta(-1)^{J-\lambda}|\vec{p},a,-\lambda\rangle; \tag{2.2}$$

we use the notation a to stand for J^η, where J (η) is the spin (parity) of the state. Notice that this result does not depend on which reference frame is used to describe the process so long as $\vec{p}$ is in the x-z plane, because Π_y commutes with Lorentz transformations in the x-z plane and with rotations about the y axis. In particular, it is true in the rest frame of R with the spin of R quantized along the z axis. This is a particularly useful operator because it commutes with the S matrix and leaves the momenta invariant; only the helicity is changed. For our present purposes only the helicity of R will be observed and so the reversal of helicity of the others is unimportant.

For the state R in its rest frame, the eigenstates of Π_y are given by

$$|\epsilon am\rangle = [\,|am\rangle + \epsilon\eta(-1)^{J-m}|a-m\rangle\,]\theta(m), \tag{2.3}$$

where

$$\theta(m) = 1/\sqrt{2}, \quad m>0$$
$$= \tfrac{1}{2}, \quad m=0$$
$$= 0, \quad m<0.$$

$\Pi_y^2 = (-1)^{2J}$ requires that $\epsilon^2 = (-1)^{2J}$ and orthogonality of the states (3) requires that $\epsilon\epsilon^* = 1$, so

$$\epsilon = \pm 1 \text{ for bosons},$$
$$\epsilon = \pm i \text{ for fermions}.$$

The eigenvalues are given by

$$\Pi_y|\epsilon am\rangle = \epsilon(-1)^{2J}|\epsilon am\rangle. \tag{2.4}$$

Notice that for bosons there is only one state for $m=0$, the one with $\epsilon = \eta(-1)^J$. The inverse of (3) is

$$|am\rangle = \sum_\epsilon [\,|\epsilon am\rangle\,\theta(m) + \eta(-1)^{J+m}\epsilon^*|\epsilon a-m\rangle\,\theta(-m)\,]. \tag{2.5}$$

The spin density matrix of R in this basis is related to the usual density matrix by

$$\begin{aligned}{}^{\epsilon\epsilon'}\rho^{aa'}_{mm'} &= \langle \epsilon am|\rho|\epsilon' a' m'\rangle \\ &= [\rho^{aa'}_{mm'} + \epsilon^*\epsilon'(-1)^{J-J'}(-1)^{m-m'}\eta\eta'\rho^{aa'}_{-m-m'} \\ &\quad + \rho^{aa'}_{-mm'}\epsilon^*(-1)^{J-m}\eta + \rho^{aa'}_{m-m'}\epsilon'(-1)^{J'-m'}\eta'] \\ &\quad \times \theta(m)\theta(m').\end{aligned} \tag{2.6}$$

If parity is conserved

$$\rho^{aa'}_{mm'} = \eta\eta'(-1)^{J-J'}(-1)^{m-m'}\rho^{aa'}_{-m-m'}. \tag{2.7}$$

Then the right-hand side of (2.6) vanishes unless $\epsilon=\epsilon'$, and ρ is diagonal in ϵ as expected. Then

$$\begin{aligned}{}^{\epsilon}\rho^{aa'}_{mm'} &= 2[\rho^{aa'}_{mm'} + \epsilon\eta'(-1)^{J'-m'}\rho^{aa'}_{m-m'}]\theta(m)\theta(m') \\ &= 2[\rho^{aa'}_{mm'} + \epsilon^*\eta(-1)^{J-m}\rho^{aa'}_{-mm'}]\theta(m)\theta(m').\end{aligned} \tag{2.8}$$

We shall call this the reflectivity density matrix. The transformation between these two bases is unitary; for bosons it is orthogonal. The inverse of (2.8) is

$$\begin{aligned}\rho^{aa'}_{mm'} = \sum_\epsilon [&{}^{\epsilon}\rho^{aa'}_{mm'}\,\theta(m)\theta(m') + \epsilon^*\eta'(-1)^{J'+m'}\,{}^{\epsilon}\rho^{aa'}_{m-m'}\,\theta(m)\theta(-m') \\ &+ \epsilon\eta(-1)^{J+m}\,{}^{\epsilon}\rho^{aa'}_{-mm'}\,\theta(-m)\theta(m') + \eta\eta'(-1)^{J-J'}(-1)^{m-m'}\,{}^{\epsilon}\rho^{aa'}_{-m-m'}\,\theta(-m)\theta(-m')].\end{aligned} \tag{2.9}$$

In matrix language, ${}^{\epsilon}\rho$ breaks up into submatrices ${}^{(+)}\rho$ and ${}^{(-)}\rho$ in block diagonal form; i.e.,

$${}^{\epsilon}\rho = \begin{pmatrix} {}^{(+)}\rho & 0 \\ 0 & {}^{(-)}\rho \end{pmatrix}. \tag{2.9'}$$

Constraints due to parity conservation are thereby explicitly and completely taken into account.

The question of angular distributions, multipole moments, and the like can be systematically studied in terms of the eigenstates of Π_y, Eq. (3). These are dealt with in the Appendix.

So far we have specified the elements of the density matrix only in terms of J, η, and m. In general, they will depend on any other internal variables that describe the state of R. For ex-

ample, there should be indices specifying the helicity and the various subenergies (if R decays into three or more particles) as well as the charge, strangeness, etc. of each of the particles that constitute R. This presents no extra complications for variables such as energy or charge, in which ρ is necessarily diagonal, or for variables such as helicity if they are not measured. They are then simply additional variables, like total energy or momentum transfer, on which ρ must depend. If one measures the helicity λ, for example of the ρ in $R \to \pi + \rho$, then a larger density matrix $\rho_{\lambda m, \lambda' m'}$ must be used. In certain favorable circumstances, the dependence on these additional variables may factor off. This seems a reasonable assumption when R consists of resonances. Then the production and decay of R separate into two steps and ρ can be taken to be the density matrix resulting from the production step. There is no clear reason why such an assumption should be valid for the background, however. For any system R consisting of a spin-0 and a spin-$\frac{1}{2}$ particle, parity conservation forces this factorization onto ρ, the reason being that when J and η are given the spin state is determined. For simplicity of presentation in developing our analysis we will usually ignore the extra variables. Since angular momentum plays very little role in the way we parametrize the matrix, it is a simple matter to include them.

Of course, angular momentum plays a central role in the *measurement* of ρ. We do not propose to go into the general question of the measurability of ρ here. This depends too much on the details of the particular experiments that can be done, the particles R decays into, the validity of factorization hypotheses, etc. We do want to discuss briefly the case where R decays into two 0^- or one 0^- and one $\frac{1}{2}^+$ particles. (See the Appendix for a more complete treatment.) If only angular distributions are measured, it is well known that only the real parts of the elements of ρ are directly measurable. One of the major uses of the positivity conditions is that they constrain the values of the unmeasured imaginary parts of ρ. If R consists of a mixture of spins then there are even more unmeasurable elements of ρ. Positivity and rank conditions can be used to constrain these as well, as we will illustrate later. The reason not all elements of ρ are measurable is easy to see: The angular distribution is given by[11]

$$I(\theta,\varphi) = \sum_{\lambda, aa' m, m'} N_J N_{J'} \rho^{aa'}_{mm'} D^{J*}_{m\lambda}(\varphi,\theta,0) \times D^{J'}_{m'\lambda}(\varphi,\theta,0) F^a_\lambda F^{a'*}_\lambda . \tag{2.10}$$

Here $\lambda = 0$ for bosonic R, $F_0 = 1$, and $\lambda = \pm\frac{1}{2}$ for fermionic R, $F_{1/2} = 1/\sqrt{2}$, $F_{-1/2} = (-1)^{J+1/2}\eta/\sqrt{2}$.

$$N_J = \left(\frac{2J+1}{4\pi}\right)^{1/2} .$$

By the usual Clebsch-Gordan series,

$$I(\theta,\varphi) = \sum \left(\frac{2L+1}{4\pi}\right)\left(\frac{2J'+1}{2J+1}\right)^{1/2} \rho^{aa'}_{mm'} (J'm', LM|Jm) \times G^{aa'}_L D^{L*}_{M0}(\varphi,\theta,0) ,$$

where

$$G^{aa'}_L = \sum_\lambda (J'\lambda L0|J\lambda) F^a_\lambda F^{a'*}_\lambda . \tag{2.11}$$

Note that $G^{aa'}_L = 0$ unless $(-1)^L = \eta\eta'$.

The moments of the distribution are

$$H(LM) = \int d\Omega\, D^L_{M0}(\varphi,\theta,0) I(\theta,\varphi) = \sum_{a,a' m, m'} \left(\frac{2J'+1}{2J+1}\right)^{1/2} \rho^{aa'}_{mm'} (J'm', LM|Jm)\, G^{aa'}_L . \tag{2.12}$$

Hermiticity of ρ implies

$$H(LM) = (-1)^M H^*(L,-M) \tag{2.13}$$

and parity conservation implies

$$H(L,-M) = (-1)^M H(LM) , \tag{2.14}$$

so the moments are all real. This prevents the determination of Imρ. In addition, because $G^{aa'}_L$ depends on J and J' the sum on the right-hand side of (2.12) cannot be inverted to allow solution for Reρ in terms of $H(LM)$ if more than one J is present. For example, in the case both 0^+ and 1^- are present, not even all the diagonal elements of ρ are measurable. Only the combinations $x = 2\rho^{11}_{11} + \rho^{11}_{00} + \rho^{00}_{00}$ and $y = \rho^{11}_{11} - \rho^{11}_{00}$ are measurable. Positivity can then be used to bound, for example, the unmeasurable s-wave element ρ^{00}_{00} both from above and below.

III. PARAMETRIZATION OF THE DENSITY MATRIX

Let us consider a collection of several spin-parity states. Let N be the dimension of the full density matrix which describes these states and their interferences. If these states are fermions, the matrices ${}^{(\pm)}\rho$ are both $(N/2)\times(N/2)$ matrices. In the case of bosons, the dimensionality of ${}^{(\pm)}\rho$ depends on the "naturality" of the bosons involved. A spin-parity series with $\eta = (-1)^J$ $[\eta = (-1)^{J+1}]$ is termed the natural (unnatural) spin-parity series. Let K_n (K_u) be the number of natural (unnatural) spin-parity states in the collection. Then the dimension of ${}^{(+)}\rho$ is $\frac{1}{2}(N + K_n - K_u)$ and that of ${}^{(-)}\rho$

is $\frac{1}{2}(N-K_n+K_u)$ because the $m=0$ states fall in ${}^{\epsilon}\rho$ with $\epsilon=\eta(-1)^J$.

The density matrix is in general given by

$$\rho^{aa'}_{mm'}=\sum_k f_{am,k}f^*_{a'm',k}\,. \tag{3.1}$$

$f_{am,k}$ denotes the production amplitude for R with spin parity a and helicity m; k denotes the unobserved variables: the spins of the other particles and, if appropriate, the production angles. As a consequence of its definition, (3.1) is a Hermitian, positive semidefinite matrix; i.e., all of its eigenvalues λ_i are real and $\lambda_i\geq 0$. If the sum in (3.1) does not involve an integration then the rank of ρ, r, is less than or equal to the number of different values the spin index k can take on. If $r<N$ there will be additional constraints on ρ.[5] In particular, N-r eigenvalues λ_i must equal zero.

The transformation to block diagonal form (2.9′) is unitary and so the submatrices ${}^{\epsilon}\rho$ are also positive. Our considerations will now be directed toward these submatrices which are subject to no additional constraints. (Rank conditions will be discussed in the next section.) Since these considerations are completely general we will discuss an arbitrary, $n\times n$ positive semidefinite, Hermitian matrix ρ with elements ρ_{ij}.

It is useful to note that if a meson system R is produced from a π or K beam by the exchange of a Regge pole of definite naturality σ, then asymptotically[12]

$$\rho^{aa'}_{m-m'}=-\sigma_R\sigma(-1)^{m'}\rho^{aa'}_{mm'}\,.$$

σ_R denotes the naturality of R:

$$\sigma_R=\eta'(-1)^{J'}\,.$$

Hence, the exchange contributes only to the block with

$$\epsilon=-\sigma\,. \tag{3.2}$$

There are many equivalent ways of stating that a matrix ρ is positive semidefinite—we will say "positive" for short—which may be found in any good book on matrices, for example that by Halmos.[13] The basic one is our definition of ρ, (3.1):

$$\rho_{ij}=\sum_k V_{ik}V^*_{jk} \tag{3.1′}$$

or equivalently

$$\rho=VV^\dagger\,. \tag{3.1″}$$

The matrices V need not be square but they can always be chosen to be so. This decomposition is not unique; any matrix $V'=VU$ where U is an arbitrary unitary matrix will also do. An equivalent definition was mentioned at the beginning of this section, viz., there exists a U such that

$$\rho=U\Lambda U^\dagger, \tag{3.3a}$$

$$\Lambda=\begin{bmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{bmatrix}, \tag{3.3b}$$

$$\lambda_i\geq 0\,. \tag{3.3c}$$

A third equivalent condition is

$$\langle x|\,\rho\,|x\rangle\geq 0 \tag{3.4}$$

for any complex vector $|x\rangle$ in the n-dimensional vector space.

A consequence of these conditions is that all of the principal minors of ρ are greater than or equal to zero. (The principal minor is the determinant of the matrix formed by removing any number of rows from the matrix along with the columns which intersect these rows on the main diagonal.) These conditions are often a useful test that can be used to quickly test the positivity of ρ. The simplest ones are the familiar conditions

$$\rho_{ii}\geq 0\,,$$

$$\rho_{ii}\rho_{jj}\geq|\rho_{ij}|^2\,.$$

Another way of checking the positivity has been given by Dalitz[3] and Minnaert.[2] We list it for convenience: Introduce the characteristic polynomial of ρ

$$\Delta(\lambda)=\det(\lambda I-\rho)$$

$$=\sum_{k=0}^{n}(-1)^k C_k\lambda^{n-k}\,. \tag{3.5}$$

Then a necessary and sufficient condition for positivity is

$$C_k\geq 0\,. \tag{3.6}$$

for all k. The simplest of these conditions are

$$C_1=\mathrm{Tr}\rho\geq 0\,,$$

$$C_n=\det\rho\geq 0\,.$$

Our adaptation of Daboul's parametrization is based on the decomposition (3.1′) or (3.1″). It uses the ambiguity of V up to multiples of a unitary matrix to bring V to superdiagonal form:

$$V_{ij}=0\,,\quad j>i \tag{3.7a}$$

$$V_{ij}=y_l e^{i\alpha_l}\,,\quad l=\tfrac{1}{2}i(i-1)+j\,,\quad j\leq i\,, \tag{3.7b}$$

with

$$y_l \geq 0\,,\ 0 \leq \alpha_l \leq 2\pi\,, \qquad \alpha_l = 0 \text{ for } l = \tfrac{1}{2}i(i+1)\,; \tag{3.8}$$

i.e., the diagonal elements are real and positive.

If ρ is normalized so that $\mathrm{Tr}\rho = 1$ there is one more constraint:

$$\sum_{l=1}^{n(n+1)/2} {y_l}^2 = 1\,. \tag{3.9}$$

This can be satisfied automatically by introducing a new set of parameters $x_1, \ldots, x_{n(n+1)/2}$ with

$$y_l = x_l \prod_{m=1}^{l-1} (1 - x_m^{\,2})^{1/2}\,, \quad l < \tfrac{1}{2}n(n+1)$$

$$0 \leq x_l < 1, \qquad x_{n(n+1)/2} = 1\,. \tag{3.10}$$

For the unnormalized matrix there are $\frac{1}{2}n(n+1)$ parameters y_l and $\frac{1}{2}n(n-1)$ parameters α_l; the sum equals n^2, the number of real parameters which determines ρ. As mentioned earlier, for an important class of reactions, only the real part of ρ can be measured in the original basis. This matrix, too, can be transformed to a parity-conserving basis by Eq. (2.8): Let $\mathrm{Re}\rho = \mathcal{R}$. Then

$$^{\epsilon}\mathcal{R}^{aa'}_{mm'} = 2[\mathcal{R}^{aa'}_{mm'} + \epsilon\eta'(-1)^{J'-m'}\mathcal{R}^{aa'}_{m-m'}] \times \theta(m)\theta(m')\,. \tag{3.11}$$

For bosons $^{\epsilon}\mathcal{R}^{aa'}_{mm'}$ is also a real, symmetric, positive matrix because the transformation is orthogonal. We can use the same parametrization for $^{\epsilon}\mathcal{R}$ as for ρ by simply setting $\alpha_l = 0$ for all l and letting y_l range over positive and negative values, except for the diagonal elements, $l = \frac{1}{2}i(i+1)$ where y_l can be chosen to be positive. There are now $\frac{1}{2}n(n+1)$ parameters, which again is exactly the number that determines $^{\epsilon}\mathcal{R}$.

For fermions the transformation (3.11) is not real and so $^{\epsilon}\mathcal{R}$ is not in general a real matrix. Thus the full set of parameters y_l and α_l as in (3.7) and (3.8) are needed. At first sight this appears to be too many parameters: n^2 for $^{+}\mathcal{R}$ and n^2 for $^{-}\mathcal{R}$. Notice, however, that

$$^{(+)}\mathcal{R}^{aa'}_{mm'} = {}^{(-)}\mathcal{R}^{aa'*}_{mm'}\,. \tag{3.12}$$

(For fermions $^{(+)}\mathcal{R}$ and $^{(-)}\mathcal{R}$ always have the same dimension.) Thus, the parameters that give $^{(+)}\mathcal{R}$ can also be used to give $^{(-)}\mathcal{R}$ by simply changing the sign of α. The total number for the full $N \times N$ matrix is then $(N/2)^2$ real parameters. A moment's thought will show that this is exactly the number of real parameters needed to describe the original matrix. (Recall that $\mathcal{R}^{aa}_{m,-m} = 0$ for fermions.)

We now give a constructive proof of the decomposition (3.1′), (3.7). There are simpler demonstrations, but we give this one because it enables one to calculate V from ρ. This can be useful in practice, since one may have a trial ρ that can serve as a starting point for the search for a more accurate ρ. First, recall that the columns of U in (3.3a) are the eigenvectors of ρ; viz., define $|u^{(k)}>$ via

$$U_{ik} = u_i^{(k)}$$

so that

$$\rho|u^{(k)}\rangle = \lambda_k|u^{(k)}\rangle\,. \tag{3.13}$$

Next define the matrix $W = U(\lambda)^{1/2}$; i.e.,

$$W_{ij} = U_{ij}(\lambda_j)^{1/2}\,, \tag{3.14}$$

and the vectors $|W^{(i)}\rangle$ as the *rows* of W:

$$W_j^{(i)} = W_{ij}\,. \tag{3.15}$$

Now construct a set of orthonormal vectors $|Z^{(i)}\rangle$ from the $|W^{(i)}\rangle$ by the Gram-Schmidt process:

$$|Z^{(1)}\rangle = \frac{|W^{(1)}\rangle}{|W^{(1)}|}\,,$$

$$|Z^{(i)}\rangle = \frac{(I - \sum_{k=1}^{i-1}|Z^{(k)}\rangle\langle Z^{(k)}|)|W^{(i)}\rangle}{|(I - \sum_{k=1}^{i-1}|Z^{(k)}\rangle\langle Z^{(k)}|)|W^{(i)}\rangle|}\,. \tag{3.16}$$

The matrix Z with

$$Z_{ij} = Z_j^{(i)}$$

is unitary, $ZZ^{\dagger} = 1$, and so

$$\rho = WW^{\dagger} = WZ^{\dagger}ZW^{\dagger}\,.$$

But by construction

$$\langle Z^{(j)}|W^{(i)}\rangle = 0\,, \quad j > i\,. \tag{3.17}$$

Consequently,

$$V = WZ^{\dagger} \tag{3.18}$$

has the form

$$V = \begin{bmatrix} V_{11} & 0 & 0 & \cdots & 0 \\ V_{21} & V_{22} & 0 & \cdots & 0 \\ V_{31} & V_{32} & V_{33} & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ V_{n1} & V_{n2} & V_{n3} & \cdots & V_{nn} \end{bmatrix}. \tag{3.19}$$

Notice that the diagonal elements $V_{ii} = \langle Z^{(i)}|W^{(i)}\rangle$ are positive real numbers. As a result there are exactly n^2 real parameters needed to describe V and hence ρ.

The same procedure works for $^{\epsilon}\mathcal{R}$ for bosons except now U is replaced by a (real) orthogonal matrix O. As a consequence the final decomposi-

tion involves a real matrix V of the form (3.19) which depends on $\frac{1}{2}n(n+1)$ real parameters.

IV. RANK CONDITIONS

If the sum over k in (3.1) involves a number of terms $r<N$, the dimension of ρ, then the rank of ρ equals r and $N-r$ of its eigenvalues must vanish. As a result, many of the positivity conditions, which are inequalities, become equalities. Thus, all principal mirrors of dimension greater than r must vanish. Equivalently,

$$C_k=0\,,\quad k>r$$

for C_k of (3.5). Obviously, these conditions will be relevant only when the sum in (3.1) does not involve integrals over production angles, missing masses, etc., though they may be approximately valid when the integrals are over small enough regions that the variation of $f_{am,k}$ can be neglected. (Even stronger rank conditions may hold if R is produced by exchange of a single Regge pole.[12])

Our parametrization can be easily adjusted to impose these conditions automatically. Suppose the rank of our $n\times n$ ρ is r. Order the eigenvalues so that

$$\lambda_{r+1}=\lambda_{r+2}=\cdots=\lambda_n=0\,. \tag{4.1}$$

Then by our construction $|W^{(i)}\rangle$ in (3.14) has only r nonzero components, $W_1^{(i)},W_2^{(i)},\ldots,W_i^{(i)}$. Hence the vectors

$$Z^{(i)}=0\,,\quad i>r\,. \tag{4.2}$$

Then by (3.18)

$$V_{ij}=\langle Z^{(j)}|W^{(i)}\rangle$$

and so

$$V_{ij}=0 \text{ for } j>r\,.$$

Thus we simply set $y_l=0$ in (3.7b) for l corresponding to $j>r$.

How do the rank conditions apply to the submatrices ${}^{\epsilon}\rho$? We will analyze here the two most common cases:

(a) $\pi N\to\pi N^*$,

(b) $\pi N\to MN$,

where N^* is any nucleon resonance and M is any boson resonance. If only the unobserved nucleon spins are summed on, the full ρ has $r=2$ in case (a) and $r=4$ in case (b). We shall now show that the nonzero eigenvalues are shared equally between ${}^{+}\rho$ and ${}^{-}\rho$.

(a) According to (3.1)

$$\rho^{aa'}_{mm'}=\sum_{\mu=\pm1/2}f_{am,\mu}f^*_{a'm',\mu}\,,$$

so

$${}^{\epsilon}\rho^{aa'}_{mm'}=\sum_{\mu=\pm1/2}[f_{am,\mu}+\epsilon(-1)^{J-m}\eta f_{a-m,\mu}]\times[f_{a'm',\mu}+\epsilon(-1)^{J'-m'}\eta' f_{a'-m',\mu}]^*\times\theta(m)\theta(m')\,. \tag{4.3}$$

But

$$\begin{aligned}f_{am,-1/2}+\epsilon(-1)^{J-m}\eta f_{a-m,-1/2}&=+(-1)^{J-m+1/2}\eta f_{a-m,1/2}\\&\quad+\epsilon(-1)^{J-m}\eta(-1)^{J+m+1/2}\eta f_{am,1/2}\\&=-\sqrt{-1}\,\epsilon[f_{am,1/2}+\epsilon(-1)^{J-m}\eta f_{a-m1/2}]\,.\end{aligned} \tag{4.4}$$

Hence

$${}^{\epsilon}\rho^{aa'}_{mm'}=2[f_{am,1/2}+\epsilon(-1)^{J-m}\eta f_{a-m,1/2}]\times[f_{a'm',1/2}+(-1)^{J'-m'}\eta' f_{a'-m',1/2}]^*\times\theta(m)\theta(m')\,, \tag{4.5}$$

and ${}^{\pm}\rho$ each have rank 1.

(b) Following the same procedure we write

$${}^{\epsilon}\rho^{aa'}_{mm'}=\sum_{\mu,\nu=\pm1/2}[f_{am\mu,\nu}+\epsilon(-1)^{J-m}\eta f_{a-m\mu,\nu}]\times[f_{a'm'\mu,\nu}+\epsilon(-1)^{J'-m'}\eta' f_{a'm'\mu,\nu}]^*\times\theta(m)\theta(m')\,.$$

Again notice that

$$\begin{aligned}f_{am\,-1/2,\nu}+\epsilon(-1)^{J-m}\eta f_{a-m-1/2,\nu}&=\epsilon[f_{am\,1/2,-\nu}+\eta\epsilon(-1)^{J-m}f_{a-m\,1/2,-\nu}](-1)^{\nu+1/2}\,,\end{aligned}$$

and since ν is summed over,

$${}^{\epsilon}\rho^{aa'}_{mm'}=2\sum_{\nu=\pm1/2}[f_{am\,1/2,\nu}+\epsilon(-1)^{J-m}\eta' f_{a-m\,1/2,\nu}]\times[f_{a'm'\,1/2,\nu}+\epsilon(-1)^{J'-m'}\eta f_{a'-m'\,1/2,\nu}]^*\,. \tag{4.6}$$

Hence ${}^{\pm}\rho$ each have rank 2.[14]

The rank conditions on $\mathfrak{R}$ are not nearly as strong because

$$\mathfrak{R}_{ij}=\sum_{k=1}^{r}V_{ik}V^*_{jk}+\sum_{k=1}^{r}V^*_{ik}V_{jk}$$

and so in general has rank $2r$.

The positivity conditions can often be combined with the rank conditions to get constraints on the unmeasured elements of ρ. The positivity conditions alone cannot give a lower bound to $|\mathrm{Im}\rho_{ij}|$, but they do give an upper bound. However, in cases where the measured rank of $\mathfrak{R}$, $r(\mathfrak{R})$, is greater than the rank of ρ, $r(\rho)$, required by the kinematics, then the imaginary part $\mathfrak{I}$ of ρ cannot vanish. The rank conditions on $\mathfrak{R}$ are easily stated: Given any two matrices A,B with $C=A+B$, then

$$r(C)\leq r(A)+r(B)\,.$$

Letting $C=\mathcal{R}$, $A=\rho$ we have

$$r(\mathcal{I}) \geq r(\mathcal{R}) - r(\rho) .$$

This theorem can be applied to the full matrices or to any of the principal submatrices formed by removing any number of rows and the corresponding columns. In the simplest case of a matrix of rank 1 one can calculate $\mathcal{I}$ up to a sign just given $\mathcal{R}$. Notice that $r(\mathcal{I})$ must be an even number because the nonzero eigenvalues of an antisymmetric Hermitian matrix necessarily occur in pairs: If λ is an eigenvalue so is $-\lambda$.

V. POLARIZED TARGET

A polarized proton target can be described by a 2×2 density matrix $r_{\mu\mu'}$. The polarization vector is given in the usual way:

$$\vec{P} = \mathrm{tr}(r\vec{\sigma}) ,$$

where $\vec{\sigma}$ represents the Pauli matrices. Explicitly,

$$P_x = r_{+1/2,-1/2} + r_{-1/2,+1/2} ,$$

$$P_y = i(r_{+1/2,-1/2} - r_{-1/2,+1/2}) ,$$

$$P_z = r_{+1/2,+1/2} - r_{-1/2,-1/2} .$$

For definiteness, let us use the s-channel helicity convention. Then, the z axis is along the proton momentum and the y axis along the production normal. Now, combining $r_{\mu\mu'}$ with the scattering amplitude f, one can write down for the density matrix of a state (or states) in the final state the following:

$$\rho^{aa'}_{mm'} \propto \sum_k f_{am,\mu k} f^*_{a'm',\mu' k} r_{\mu\mu'} . \tag{5.1a}$$

Or, in terms of the polarization vector, one can write

$$\rho = \rho^0 + \rho^n \vec{P}\cdot\hat{n} + \rho^{\parallel}\vec{P}\cdot\hat{p} + \rho^{\perp}\vec{P}\cdot\hat{n}\times\hat{p} . \tag{5.1b}$$

$\hat{n}$ denotes the normal to the production plane, $\hat{p}$ the unit vector in the direction of the proton momentum in the c.m. system. ρ^0 is the unpolarized density matrix discussed before.

It is a simple matter to see that the parity conditions analogous to (2.7) are

$$(\rho^n)^{aa'}_{mm'} = \eta\eta'(-1)^{J-J'}(-1)^{m-m'}(\rho^n)^{aa'}_{-m,-m'} , \tag{5.2a}$$

$$(\rho^{\parallel,\perp})^{aa'}_{mm'} = -\eta\eta'(-1)^{J-J'}(-1)^{m-m'}(\rho^{\parallel,\perp})^{aa'}_{-m,-m'} . \tag{5.2b}$$

Because of these parity conditions, the transformation (2.6) will bring ρ^n to block diagonal form but $\rho^{\perp}$ and $\rho^{\parallel}$ will be off-diagonal in ϵ. Then, for example, in production of a meson resonance from a π or a K by exchange of Regge poles, $\rho^{\perp}$ and $\rho^{\parallel}$ get nonzero contributions only from interference between poles of opposite naturality, while ρ^0 and ρ^n depend only on interference between poles of like naturality.

Let us restrict our attention now to the common case where the polarization is normal to the plane. All of our previous results then go through: ρ is positive and can be brought to block-diagonal ${}^{\epsilon}\rho$ form by the unitary transformation (2.6); each ${}^{\epsilon}\rho$ can be parametrized as before. However, now the rank conditions can be more restrictive. Consider the two simple cases discussed in Sec. IV:

(a) $\pi N \to \pi N^*$. In this case

$$\begin{aligned}
&(\rho^0)^{aa'}_{mm'} = (f_{am,1/2} f^*_{a'm',1/2} + f_{am,-1/2} f^*_{a'm',-1/2}) ,\\
&(\rho^n)^{aa'}_{mm'} = -i(f_{am,1/2} f^*_{a'm',-1/2} - f_{am,-1/2} f^*_{a'm',1/2}) ,\\
&(\rho^0 \pm \rho^n)^{aa'}_{mm'} = (f_{am,1/2} \pm i f_{am,-1/2})(f_{a'm',1/2} \pm i f_{a'm',-1/2})^* .
\end{aligned} \tag{5.3}$$

Hence $\rho^0 \pm \rho^n$ each have rank 1. Furthermore

$$(\rho^0 \pm \rho^n)^{aa'}_{m-m'} = \mp i(-1)^{J'-m'}\eta'(\rho^0 \pm \rho^n)^{aa'}_{mm'} , \tag{5.4}$$

so when $\rho^0 \pm \rho^n$ is reduced to block-diagonal form it has only one nonzero block, that for $\epsilon = \pm 1$, and that block has rank 1. Hence, as pointed out before, the imaginary part of $\rho^0 \pm \rho^n$ can be calculated, up to a sign, given the real part of $\rho^0 \pm \rho^n$, which can be obtained by varying the degree of polarization.

(b) $\pi N \to MN$. In addition to Eq. (3.6) for ${}^{\epsilon}\rho^0$ we need

$$\begin{aligned}
({}^{\epsilon}\rho^n)^{aa'}_{mm'} = &-2i[f_{am\,1/2,1/2} + \epsilon(-1)^{J-m}\eta f_{a-m\,1/2,1/2}][f_{am'\,1/2,-1/2} + \epsilon(-1)^{J'-m'}\eta' f_{a'-m'\,1/2,-1/2}]^*\\
&+2i[f_{am\,1/2,-1/2} + \epsilon(-1)^{J-m}\eta f_{a-m\,1/2,-1/2}][f_{a'm'\,1/2,1/2} + \epsilon(-1)^{J'-m'}\eta' f_{a'-m'\,1/2,1/2}]^*\theta(m)\theta(m') .
\end{aligned} \tag{5.5}$$

Combining (4.6) and (5.5) we obtain

$$\begin{aligned}
({}^{\epsilon}\rho^0 \pm {}^{\epsilon}\rho^n)^{aa'}_{mm'} = 2\theta(m)\theta(m')\{&[f_{am\,1/2,1/2} + \epsilon(-1)^{J-m}\eta f_{a-m\,1/2,1/2}] \pm i[f_{am\,1/2,-1/2} + \epsilon(-1)^{J-m}\eta f_{a-m\,1/2,-1/2}]\}\\
\times\{&[f_{a'm'\,1/2,1/2} + \epsilon(-1)^{J'-m'}\eta' f_{a'-m'\,1/2,1/2}] \pm i[f_{a'm'\,1/2,-1/2} + \epsilon(-1)^{J'-m'}\eta' f_{a'-m'\,1/2,-1/2}]\}^* .
\end{aligned} \tag{5.6}$$

This explicitly shows that ${}^{\epsilon}\rho^{0} \pm {}^{\epsilon}\rho^{\eta}$ are each of rank 1. Once again the imaginary parts of these matrices can be calculated (up to a sign) given only the real parts.

For a further discussion of these points, see Ref. 15.

VI. EXAMPLES

We give here a few simple but, in practice, important examples to illustrate the methods outlined in the previous sections.

A. Mixture of spin-parity states 0^+ and 1^-

Let us use the notations s and p to denote spin-pairty states 0^+ and 1^-. Since both spin-parity states belong to natural series, the matrices ${}^{(+)}\rho$ and ${}^{(-)}\rho$ have dimensions 3 and 1, respectively. They may be written

$$ {}^{(+)}\rho = \begin{pmatrix} a & d^* & e^* \\ d & b & f^* \\ e & f & c \end{pmatrix}, \quad {}^{(-)}\rho = (g) \tag{6.1} $$

where

$$ \begin{aligned} a &= {}^{(+)}\rho^{ss}_{00} = \rho^{ss}_{00}, \\ b &= {}^{(+)}\rho^{pp}_{00} = \rho^{pp}_{00}, \\ c &= {}^{(+)}\rho^{pp}_{11} = \rho^{pp}_{11} - \rho^{pp}_{1-1}, \\ d &= {}^{(+)}\rho^{ps}_{00} = \rho^{ss}_{00}, \\ e &= {}^{(+)}\rho^{ps}_{10} = \sqrt{2}\,\rho^{ps}_{10}, \\ f &= {}^{(+)}\rho^{pp}_{10} = \sqrt{2}\,\rho^{pp}_{10}, \\ g &= {}^{(-)}\rho^{pp}_{11} = \rho^{pp}_{11} + \rho^{pp}_{1-1}. \end{aligned} $$

From non-negativeness of the principal minors, we obtain

$$ a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad g \geq 0, \tag{6.2a} $$

$$ ab \geq |d|^2, \quad ac \geq |e|^2, \quad bc \geq |f|^2, \tag{6.2b} $$

$$ abc + 2\,\mathrm{Re}(dfe^*) \geq a|f|^2 + b|e|^2 + c|d|^2. \tag{6.2c} $$

These inequalities then express fully the positivity conditions. Suppose the states s and p are produced in a reaction $\pi N \to RN$ (with R standing for both s and p). Then, the dimensionality r of the unobserved spin space is 2, so that the Eberhard-Good theorem amounts to taking the equality sign in (6.2c). It should be noted that inequalities (6.2) contain the positivity constraints on pure spin-parity states. As for the s state, $a \geq 0$ implies that $\rho^{ss}_{00} \geq 0$. The physical domain for the p state is contained in the inequalities $b \geq 0$, $c \geq 0$, $g \geq 0$, and $bc \geq |f|^2$. They lead to the well-known relations

$$ \begin{aligned} &\rho^{pp}_{11} > |\rho^{pp}_{1-1}|, \\ &\rho^{pp}_{00}(\rho^{pp}_{11} - \rho^{pp}_{1-1}) > 2|\rho^{pp}_{10}|^2. \end{aligned} \tag{6.3} $$

In many decay modes, such as those into two spinless particles, only the real part of ρ can be determined owing to parity conservation in the decay. The resulting density matrices $\mathrm{Re}^{(+)}\rho$ and $\mathrm{Re}^{(-)}\rho$ are again "positive"; however, the dimensionality r of the unobserved spin space increases to 4 (for the reaction $\pi N \to RN$), so that the Eberhard-Good theorem imposes no constraint. The positivity conditions in this case are obtained by substituting the real part for all the complex parameters appearing in (6.2). Equation (6.2a) remains the same, while (6.2b) now reads

$$ ab \geq d_R^2, \quad ac \geq e_R^2, \quad bc \geq f_R^2 \tag{6.2d} $$

and (6.2c) reads

$$ abc + 2d_R f_R e_R \geq a f_R^2 + b e_R^2 + c d_R^2 \tag{6.2e} $$

where the subscripts denote the real part.

In some model calculations, such as those involving the mass-dependent angular analysis,[16] it is possible to determine both the real and the imaginary part of the s-p interference terms (d and e), while only the real part f for the pure p state can be measured. In this case, the measurable part of the density matrix ${}^{(+)}\rho$ is not necessarily positive; the only way to obtain the appropriate positivity conditions is to explore the relations (6.2c) in the space of the unmeasured quantity f_I and find the extremum; from (6.2b)

$$ ab \geq |d|^2, \quad ac \geq |e|^2, \quad bc \geq f_R^2 \tag{6.2f} $$

and from (6.2c)

$$ abc + 2p_R(de^*)_R + (de^*)_I^2 \geq a f_R^2 + b|e|^2 + c|d|^2. \tag{6.2g} $$

Next, we give a parametrization of ${}^{(+)}\rho$ which guarantees that it remains within the physical domain. Let

$$ {}^{(+)}\rho_{ij} = \sum_{k=1}^{3} V_{ik} V^*_{jk} = \langle V_j | V_i \rangle \quad (i,j = 1,2,3), \tag{6.4} $$

where $|V_i\rangle = (V_{i1}, V_{i2}, V_{i3})$. Explicitly,

$$ \begin{aligned} &a = \langle V_1 | V_1 \rangle, \quad b = \langle V_2 | V_2 \rangle, \quad c = \langle V_3 | V_3 \rangle, \\ &d = \langle V_1 | V_2 \rangle, \quad e = \langle V_1 | V_3 \rangle, \quad f = \langle V_2 | V_3 \rangle. \end{aligned} \tag{6.5} $$

Then,

$$ \begin{aligned} |V_1\rangle &= (y_1, 0, 0), \\ |V_2\rangle &= (y_2 e^{i\alpha_2}, y_3, 0), \\ |V_3\rangle &= (y_4 e^{i\alpha_4}, y_5 e^{i\alpha_5}, y_6). \end{aligned} \tag{6.6} $$

Note that, if $r = 2$, the vectors $|V_i\rangle$ are defined in

2-dimensional space and $y_6 = 0$ (Eberhard-Good theorem).

With $g \equiv y_7$, the trace condition

$$\mathrm{tr}^{(+)}\rho + \mathrm{tr}^{(-)}\rho = 1$$

translates into

$$\sum_{i=1}^{7} y_i^2 = 1 .$$

To guarantee this constraint, we reexpress the y_i's in terms of the new variables x_i:

$$\begin{aligned} y_1 &= x_1 , \\ y_i &= \prod_{k=1}^{i-1} (1 - x_k^2)^{1/2} x_i \quad (i = 2, 6) , \\ y_7 &= \prod_{k=1}^{6} (1 - x_k^2)^{1/2} , \end{aligned} \tag{6.7}$$

where $0 \leq x_i \leq 1$ for $i = 1, 6$.

Thus, we have found a new set of 9 independent variables, x_i $(i = 1, 6)$, α_2, α_4, and α_5, which guarantee that ${}^{(\pm)}\rho$ remains within the physical domain. If only the real part of ${}^{(\pm)}\rho$ can be determined, the parametrization involves merely setting $\alpha_2 = \alpha_4 = \alpha_5 = 0$ and requiring that $-1 \leq x_i \leq +1$ $(i = 2, 4, 5)$ and $0 \leq x_i \leq +1$ $(i = 1, 3, 6)$.

B. Mixture of spin-parity states 0^- and 1^-

We take up this example in order to illustrate the case of spin-parity states with mixed naturality. We shall again use the notations s and p for 0^- and 1^-. The reflectivity density matrices have the form

$${}^{(+)}\rho = \begin{pmatrix} a & e^* \\ e & b \end{pmatrix} , \quad {}^{(-)}\rho = \begin{pmatrix} c & f^* \\ f & d \end{pmatrix} , \tag{6.8}$$

where

$$\begin{aligned} a &= {}^{(+)}\rho^{pp}_{00} = \rho^{pp}_{00} \\ b &= {}^{(+)}\rho^{pp}_{11} = \rho^{pp}_{11} - \rho^{pp}_{1\,-1} , \\ c &= {}^{(-)}\rho^{ss}_{00} = \rho^{ss}_{00} , \\ d &= {}^{(-)}\rho^{pp}_{11} = \rho^{pp}_{11} + \rho^{pp}_{1\,-1} \\ e &= {}^{(+)}\rho^{pp}_{10} = \sqrt{2}\,\rho^{pp}_{10} , \\ f &= {}^{(-)}\rho^{ps}_{10} = \sqrt{2}\,\rho^{ps}_{10} . \end{aligned}$$

The positivity conditions assume the form

$$a \geq 0 , \quad b \geq 0 , \quad c \geq 0 , \quad d \geq 0 , \tag{6.9a}$$

$$ab \geq |e|^2 , \quad cd \geq |f|^2 . \tag{6.9b}$$

If only the real part of ρ is measurable, we may simply replace the absolute squares in (6.9b) by $e_R^{\,2}$ and $f_R^{\,2}$.

We can parametrize ${}^{(+)}\rho$ and ${}^{(-)}\rho$ by setting

$$\begin{aligned} a &= \langle V_1 | V_1 \rangle , \quad b = \langle V_2 | V_2 \rangle , \quad e = \langle V_1 | V_2 \rangle , \\ c &= \langle W_1 | W_1 \rangle , \quad d = \langle W_2 | W_2 \rangle , \quad f = \langle W_1 | W_2 \rangle , \end{aligned} \tag{6.10}$$

where

$$\begin{aligned} |V_1\rangle &= (y_1, 0) , \quad |V_2\rangle = (y_2 e^{i\alpha_2}, y_3) , \\ |W_1\rangle &= (y_4, 0) , \quad |W_2\rangle = (y_5 e^{i\alpha_5}, y_6) . \end{aligned} \tag{6.11}$$

From the trace condition we have the constraint

$$\sum_{i=1}^{6} y_i^2 = 1 ,$$

so that in terms of the new variables x_i

$$\begin{aligned} y_1 &= x_1 , \\ y_i &= \prod_{k=1}^{i-1} (1 - x_k^2)^{1/2} x_i \, (i = 2, 5) , \\ y_6 &= \prod_{k=1}^{5} (1 - x_k^2)^{1/2} , \end{aligned} \tag{6.12}$$

where $0 \leq x_1 \leq 1$ $(i = 1, 5)$. Therefore, a set of seven independent parameters x_i $(i = 1, 5)$, α_2, and α_5 describes fully the space of ${}^{(+)}\rho$ and ${}^{(-)}\rho$ and guarantees that they remain within the physical domain

C. Mixture of spin-parity states 0^+ and 2^+

We shall use the notations s and d for 0^+ and 2^+. Then the reflectivity density matrices have the form

$${}^{(+)}\rho = \begin{bmatrix} a & e^* & f^* & h^* \\ e & b & g^* & p^* \\ f & g & c & v^* \\ h & p & v & d \end{bmatrix} , \quad {}^{(-)}\rho = \begin{pmatrix} l & w^* \\ w & m \end{pmatrix} , \tag{6.13}$$

where

$$\begin{aligned} a &= {}^{(+)}\rho^{ss}_{00} = \rho^{ss}_{00} , \\ b &= {}^{(+)}\rho^{dd}_{00} = \rho^{dd}_{00} , \\ c &= {}^{(+)}\rho^{dd}_{11} = \rho^{dd}_{11} - \rho^{dd}_{1\,-1} , \\ d &= {}^{(+)}\rho^{dd}_{22} = \rho^{dd}_{22} + \rho^{dd}_{2\,-2} , \\ e &= {}^{(+)}\rho^{ds}_{00} = \rho^{ds}_{00} , \\ f &= {}^{(+)}\rho^{ds}_{10} = \sqrt{2}\,\rho^{ds}_{10} , \\ g &= {}^{(+)}\rho^{dd}_{10} = \sqrt{2}\,\rho^{dd}_{10} , \\ h &= {}^{(+)}\rho^{ds}_{20} = \sqrt{2}\,\rho^{ds}_{20} , \\ p &= {}^{(+)}\rho^{dd}_{20} = \sqrt{2}\,\rho^{dd}_{20} , \\ v &= {}^{(+)}\rho^{dd}_{21} = \rho^{dd}_{21} - \rho^{dd}_{2\,-1} , \\ l &= {}^{(-)}\rho^{dd}_{11} = \rho^{dd}_{11} + \rho^{dd}_{1\,-1} , \\ m &= {}^{(-)}\rho^{dd}_{22} = \rho^{dd}_{22} - \rho^{dd}_{2\,-2} , \\ w &= {}^{(-)}\rho^{dd}_{21} = \rho^{dd}_{21} + \rho^{dd}_{2\,-1} . \end{aligned}$$

The positivity conditions are obtained by setting all the principal minors of ${}^{(\pm)}\rho$ non-negative,

$$a \geq 0,\ b \geq 0,\ c \geq 0,\ d \geq 0,\ l \geq 0,\ m \geq 0, \tag{6.14a}$$

$$\begin{vmatrix} a & e^* \\ e & b \end{vmatrix} \geq 0, \quad \begin{vmatrix} a & f^* \\ f & c \end{vmatrix} \geq 0, \quad \begin{vmatrix} a & h^* \\ h & d \end{vmatrix} \geq 0, \tag{6.14b}$$

$$\begin{vmatrix} b & g^* \\ g & c \end{vmatrix} \geq 0, \quad \begin{vmatrix} b & p^* \\ p & d \end{vmatrix} \geq 0, \quad \begin{vmatrix} c & v^* \\ v & d \end{vmatrix} \geq 0, \quad \begin{vmatrix} l & w^* \\ w & m \end{vmatrix} \geq 0,$$

$$\begin{vmatrix} a & e^* & f^* \\ e & b & g^* \\ f & g & c \end{vmatrix} \geq 0, \quad \begin{vmatrix} a & e^* & h^* \\ e & b & p^* \\ h & p & d \end{vmatrix} \geq 0, \quad \begin{vmatrix} b & g^* & p^* \\ g & c & v^* \\ p & v & d \end{vmatrix} \geq 0, \tag{6.14c}$$

$$\begin{vmatrix} a & e^* & f^* & h^* \\ e & b & g^* & p^* \\ f & g & c & v^* \\ h & p & v & d \end{vmatrix} \geq 0. \tag{6.14d}$$

If $r = 2$, inequalities in (6.14c) and (6.14d) become equalities. If only $\mathrm{Re}\,{}^{(+)}\rho$ are measurable, complex variables in (6.14b), (6.14c), and (6.14d) are replaced by their real parts; if $r = 4$ in this case, inequalities in (6.14) remain inequalities (no Eberhard-Good theorem). Note that (6.14a), the fourth to seventh inequalities of (6.14b), and the third inequality of (6.14c) together constitute the positivity conditions for the pure states s and d.

We give, as an example, the parametrizations of ${}^{(\pm)}\rho$ when $r = 2$. Let us write

$$\begin{aligned} {}^{(+)}\rho_{ij} &= \langle V_j | V_i \rangle \quad (i,j = 1,2,3,4), \\ {}^{(-)}\rho_{ij} &= \langle W_j | W_i \rangle \quad (i,j = 1,2), \end{aligned} \tag{6.15}$$

where the vectors $|V_i\rangle$ and $|W_i\rangle$ are defined in 2-dimensional space. Then,

$$\begin{aligned} |V_1\rangle &= (y_1, 0), \\ |V_2\rangle &= (y_2 e^{i\alpha_2}, y_3), \\ |V_3\rangle &= (y_4 e^{i\alpha_4}, y_5 e^{i\alpha_5}), \\ |V_4\rangle &= (y_6 e^{i\alpha_6}, y_7 e^{i\alpha_7}), \\ |W_1\rangle &= (y_8, 0), \\ |W_2\rangle &= (y_9 e^{i\alpha_9}, y_{10}). \end{aligned} \tag{6.16}$$

In order to satisfy the trace condition,

$$\sum_{i=1}^{10} y_i^2 = 1,$$

we define

$$\begin{aligned} y_1 &= x_1, \\ y_i &= \prod_{k=1}^{i-1} (1 - x_k^2)^{1/2} x_i \quad (i = 2, 9), \\ y_{10} &= \prod_{k=1}^{9} (1 - x_k^2)^{1/2}, \end{aligned} \tag{6.17}$$

where $0 \leq x_i \leq 1$ $(i = 1, 9)$. Therefore, the desired parametrization is accomplished with a set of 15 parameters $[x_i\ (i = 1, 9),\ \alpha_2,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7,$ and $\alpha_9]$. If $\mathrm{Im}\,{}^{(\pm)}\rho$ is not measurable, we set the α_i's to zero and let x_i range from -1 to $+1$, except x_1, x_3, and x_8 which are kept within the range 0 to 1.

D. Mixture of spin-parity states $\frac{1}{2}^+$ and $\frac{3}{2}^+$ (or pure state $\frac{5}{2}^+$).

A mixture of states involving fermions is somewhat simpler to deal with, for the analysis does not depend on the naturality of spin-parity states as was the case with bosons. The reflectivity density matrices are 3×3 matrices for either a mixture of spins $\frac{1}{2}$ and $\frac{3}{2}$ or a pure $\frac{5}{2}$ spin state regardless of the intrinsic parities involved; we have merely chosen here the positivity-parity states as an example.

Let us use the notations s, p, and d to stand for $\frac{1}{2}^+$, $\frac{3}{2}^+$, and $\frac{5}{2}^+$. Our first task is to explore the dimensionality r of the unobserved spin space. Consider a reaction $\pi N \to \pi R$, where R stands for either s and p or d. Then we see that $r = 1$. Next, consider $\pi N \to VR$, where V stands for either a vector or a pseudovector meson. Then we have $r = 3$.

The reflectivity density matrices ${}^{(\pm)}\rho$ have the form

$${}^{(+)}\rho = \begin{pmatrix} a & d^* & e^* \\ d & b & f^* \\ e & f & c \end{pmatrix}, \quad {}^{(-)}\rho = \begin{pmatrix} g & p^* & v^* \\ p & h & u^* \\ v & u & e \end{pmatrix}, \tag{6.18}$$

where, for a mixture of s and p,

$$\begin{aligned} a &= {}^{(+)}\rho_{11}^{ss} = \rho_{11}^{ss} + i\rho_{1\,-1}^{ss}, \\ b &= {}^{(+)}\rho_{11}^{pp} = \rho_{11}^{pp} - i\rho_{1\,-1}^{pp}, \\ c &= {}^{(+)}\rho_{33}^{pp} = \rho_{33}^{pp} + i\rho_{3\,-3}^{pp}, \\ d &= {}^{(+)}\rho_{11}^{ps} = \rho_{11}^{ps} + i\rho_{1\,-1}^{ps}, \\ e &= {}^{(+)}\rho_{31}^{ps} = \rho_{31}^{ps} + i\rho_{3\,-1}^{ps}, \\ f &= {}^{(+)}\rho_{31}^{pp} = \rho_{31}^{pp} - i\rho_{3\,-1}^{pp} \end{aligned}$$

and the elements of ${}^{(-)}\rho$ are the same as those of ${}^{(+)}\rho$ with the sign of the second terms reversed. (We use the convention whereby the subscripts of ρ stand for twice the z component of spin.) In the case of a pure spin state d we have, in a compact

form,

$$^{(\pm)}\rho^{dd}_{ij} = \rho^{dd}_{nm} \pm i(-)^{(J-m)/2}\rho^{dd}_{n-m} \quad (i,j=1,2,3),$$

where $n=2i-1$ and $m=2j-1$.

The positivity conditions are

$$a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad g \geq 0, \quad h \geq 0, \quad e \geq 0, \tag{6.19a}$$

$$ab \geq |d|^2, \quad ac \geq |e|^2, \quad bc \geq |f|^2,$$
$$gh \geq |p|^2, \quad ge \geq |v|^2, \quad he \geq |u|^2, \tag{6.19b}$$

$$\begin{vmatrix} a & d^* & * \\ d & b & f^* \\ e & f & c \end{vmatrix} \geq 0, \quad \begin{vmatrix} g & p^* & v^* \\ p & h & u^* \\ v & u & e \end{vmatrix} \geq 0. \tag{6.19c}$$

[If $r=1$, "equals" signs prevail in (6.19b) and (6.19c).] The positivity of a pure state s is contained in $a \geq 0$ and $g \geq 0$, which may be written

$$\rho^{ss}_{11} \geq |\rho^{ss}_{1-1}|. \tag{6.20}$$

The positivity conditions for a pure p state are, from (6.19a) and (6.19b),

$$\rho^{pp}_{11} \geq |\rho^{pp}_{1-1}|,$$
$$\rho^{pp}_{33} \geq |\rho^{pp}_{3-3}|, \tag{6.21}$$
$$(\rho^{pp}_{11} \pm i\rho^{pp}_{1-1})(\rho^{pp}_{33} \mp i\rho^{pp}_{3-3}) \geq |\rho^{pp}_{31} \pm i\rho^{pp}_{3-1}|^2.$$

If only $\mathrm{Re}\rho^{pp}_{nm}$ are measurable, we have, noting that ρ^{pp}_{nn} (ρ^{pp}_{n-n}) is purely real (imaginary),

$$\rho^{pp}_{11} \geq 0,$$
$$\rho^{pp}_{33} \geq 0, \tag{6.22}$$
$$\rho^{pp}_{11}\rho^{pp}_{33} \geq (\mathrm{Re}\rho_{31})^2 + (\mathrm{Re}\rho_{3-1})^2.$$

These relations follow from (6.21) if ρ^{pp}_{nm} is replaced by $\mathrm{Re}\rho^{pp}_{nm}$.

Assuming $r=1$, we now turn to the parametrization of $^{(\pm)}\rho$. Again, we set

$$^{(+)}\rho_{ij} = \langle V_j | V_i \rangle, \quad ^{(-)}\rho_{ij} = \langle W_j | W_i \rangle \quad (i,j=1,2,3), \tag{6.23}$$

where $|V_i\rangle$ and $|W_i\rangle$ are vectors defined in one-dimensional space,

$$|V_1\rangle = (y_1), \quad |W_1\rangle = (y_4)$$
$$|V_2\rangle = (y_2 e^{i\alpha_2}), \quad |W_2\rangle = (y_5 e^{i\alpha_5}) \tag{6.24}$$
$$|V_3\rangle = (y_3 e^{i\alpha_3}), \quad |W_3\rangle = (y_6 e^{i\alpha_6})$$

and the x_i's which define y_i's are identical to those given in (6.12). Therefore, the parametrization of $^{(\pm)}\rho$ requires a total of nine variables, x_i ($i=1,5$), α_2, α_3, α_5, and α_6. Suppose now that only $\mathrm{Re}\rho^{aa'}_{nn}$ can be measured. We have to deal with $^{(+)}\mathcal{R}$ only, since $^{(-)}\mathcal{R}$ is related to $^{(+)}\mathcal{R}$. Note that $^{(+)}\mathcal{R}$ is still complex but that $r=2$ in this case. Therefore,

$$|V_1\rangle = (y_1, 0),$$
$$|V_2\rangle = (y_2 e^{i\alpha_2}, y_3), \tag{6.25}$$
$$|V_3\rangle = (y_4 e^{i\alpha_4}, y_5 e^{i\alpha_5}),$$

and the corresponding x_i's are as given in (3.10). Thus, the parametrization of $^{(\pm)}\rho$ in this case involves a set of seven parameters, x_i ($i=1,4$), α_2, α_4, and α_5.

APPENDIX: ANGULAR DISTRIBUTION

In order to illustrate how density matrices come into the expressions for angular distributions, we will take a concrete example of a resonance R (produced via $AB \to RC$) with spin parity $a=(J,\eta)$ and a background with spin parity a', both of which decay into two pions plus a particle with spin parity s^{η_s}. We choose this example because it encompasses not only a wide variety of known resonance decays but also simpler two-body decays with certain parameters set to zero.[17]

Let $R=(\alpha,\beta,\gamma)$ be the Euler angles which describe the orientation of a 3-particle system $(S+\pi+\pi)$ in the helicity rest frame (z axis along $-C$ and y axis along the production normal). The angular distribution in R can be broken up into terms corresponding to contributions from spin parity a, a', and the interference between them, viz.,

$$I(R) = \sum_{a,a'} I^{aa'}(R) \tag{A1}$$

with the normalization

$$\int I(R)dR = 1. \tag{A2}$$

The distribution $I^{aa'}(R)$ is given in terms of the density-matrix elements as follows[16]:

$$I^{aa'}(R) = N_J N_{J'} \rho^{aa'}_{mm'} D^{J*}_{m\mu}(R)\, D^{J'}_{m',\mu'}(R)\, F^{a}_{\mu\lambda} F^{a'*}_{\mu'\lambda}, \tag{A3}$$

where λ is the helicity of s and summation is implied over repeated indices. N_J is a normalization constant,

$$N_J = \left(\frac{2J+1}{8\pi^2}\right)^{1/2}, \tag{A4}$$

and $F^{a}_{\mu\lambda}$ is the coupling constant with μ signifying the z component of spin along a body-fixed z axis (thus μ is a rotational invariant). If it is chosen to be along the normal to the decay plane of $(s+\pi+\pi)$, parity conservation in the decay demands that

$$F^{a}_{\mu\lambda} = \eta\eta_s(-)^{s+\mu} F^{a}_{\mu-\lambda}. \tag{A5}$$

If R decays into a two-body system $(s+\pi)$, the

corresponding angular distribution is similar to that that obtained by integrating (A3) over the angle γ. Thus,

$$I^{aa'}(\Omega) = 2\pi N_J N_{J'} \rho^{aa'}_{mm'} D^{J*}_{m\lambda}(\Omega) D^{J'}_{m'\lambda}(\Omega) F^a_\lambda F^{a'*}_\lambda , \tag{A6}$$

where $\Omega = (\alpha, \beta, 0)$. Parity conservation in the decay demands that

$$F^a_\lambda = \eta\eta_s(-)^{J-s-1} F^a_{-\lambda} . \tag{A7}$$

The distribution (A6) shows that two-body decays into $(s+\pi)$ can be treated by effecting the following substitutions in (A3): $R \to \Omega$, $N_J \to (2\pi)^{1/2} N_J$, μ or $\mu' \to \lambda$, and $F^a_{\mu\lambda}$ or $F^a_{\mu'\lambda} \to F^a_\lambda$. For this reason, we shall henceforth consider in most instances only the three-body decays into $(s+\pi+\pi)$ for the sake of simplicity.

It is convenient for certain applications to expand the angular distributon (A3) in terms of the moments. A moment, which can be measured experimentally,

$$H(LMN) = \langle D^L_{MN}(R) \rangle = \int dR\, I(R)\, D^L_{MN}(R) , \tag{A8}$$

is a sum of "partial moments" which specify contributions from each spin-parity state (or the interference between different spin-parity states), viz.,

$$H(LMN) = \sum_{a,a'} H^{aa'}(LMN) , \tag{A9}$$

where

$$H^{aa'}(LMN) = t^{aa'*}_{LM} G^{aa'}_{LN} . \tag{A10}$$

The first factor is known as the multipole parameter, which is given by

$$t^{aa'*}_{LM} = \left(\frac{2J'+1}{2J+1}\right)^{1/2} \sum_{mm'} \rho^{aa'}_{mm'} (J'm'LM|Jm) , \tag{A11a}$$

with the inverse given by

$$\rho^{aa'}_{mm'} = \sum_{LM} \left(\frac{2L+1}{(2J+1)(2J'+1)}\right)^{1/2} \times t^{aa'*}_{LM} (J'm'LM|Jm) , \tag{A11b}$$

while the second factor is related to the coupling constant,

$$G^{aa'}_{LN} = \sum_{\mu\mu'} (J'\mu'LN|J\mu) \sum_\lambda F^a_{\mu\lambda} F^{a'*}_{\mu'\lambda} . \tag{A12a}$$

For the two-body decay $(s+\pi)$,

$$G^{aa'}_L = \sum_\lambda (J'\lambda L0|J\lambda) F^a_\lambda F^{a'*}_\lambda . \tag{A12b}$$

The "partial" angular distribution (A3) now has a simple expansion in terms of the partial moments,

$$I^{aa'}(R) = \sum_L \left(\frac{2L+1}{8\pi^2}\right) \sum_{MN} H^{aa'}(LMN)\, D^{L*}_{MN}(R) . \tag{A13}$$

Let us at this point consider the consequences of parity conservation. The conservation in the production process implies, from parity conservation,

$$t^{aa'}_{LM} = \eta\eta'(-)^{L+M} t^{aa'}_{L,-M} \tag{A14a}$$

and, from the hermicity of ρ,

$$t^{aa'}_{LM} = \left(\frac{2J'+1}{2J+1}\right)^{1/2} (-1)^{J-J'} (-1)^M t^{a'a*}_{L,-M} , \tag{A14b}$$

while for the decay process, from (A5),

$$G^{aa'}_{LN} = \eta\eta'(-)^N G^{aa'}_{LN} , \tag{A15a}$$

so that N is even (odd) if $\eta\eta'$ is even (odd). Under the interchange of a and a',

$$G^{aa'}_{LN} = \left(\frac{2J+1}{2J'+1}\right)^{1/2} (-1)^{J-J'} (-1)^N G^{a'a*}_{L,-N} . \tag{A15b}$$

If the decay is into two bodies $(s+\pi)$, from (A7) and (A12b),

$$G^{aa'}_L = \eta\eta'(-)^L G^{aa'}_L \tag{A15c}$$

under the interchange of a and a', $G^{aa'}_L$ transforms as shown in (A15b) with $N=0$.

We have so far treated the problem of constructing an expression for angular distribution (which includes interference) in terms of the conventional density-matrix elements. Our task now is to treat the problem again starting from reflectivity density-matrix elements. For this purpose, it is necessary to define a new rotation function, with $\kappa \equiv \epsilon\eta$,

$${}^\kappa D^J_{m\mu}(R) = \langle \epsilon a m | R | a \mu \rangle , \tag{A16}$$

which is related to the conventional D functions via [see (2.3)]

$${}^\kappa D^J_{m\mu}(R) = [D^J_{m\mu}(R) + \kappa^*(-)^{J-m} D^J_{-m\mu}(R)]\, \theta(m) , \tag{A17}$$

and the complex conjugation is given by

$${}^\kappa D^{J*}_{m\mu}(R) = \kappa(-)^{J+\mu}\, {}^\kappa D^J_{m-\mu}(R) . \tag{A17'}$$

Although they are not representations of the rotational group, the new D functions have many similar properties: They are orthonormal,

$$\int dR\, {}^{\kappa_1}D^{J_1}_{m_1\mu_1}{}^{*}(R)\, {}^{\kappa_2}D^{J_2}_{m_2\mu_2}(R) = \frac{8\pi^2}{2J_1+1}\delta_{J_1J_2}\delta_{m_1m_2}\delta_{\mu_1\mu_2}\delta_{\kappa_1\kappa_2}\ . \tag{A18}$$

From the fact that reflection eigenstates form a complete orthonormal set, one can easily show that

$${}^{\kappa_1}D^{J_1}_{m_1\mu_1}(R)\, {}^{\kappa_2}D^{J_2}_{m_2\mu_2}(R) = \sum_{\kappa_3 J_3 m_3 \mu_3} \langle \kappa_1 J_1 m_1 \kappa_2 J_2 m_2 \,|\, \kappa_3 J_3 m_3\rangle\, (J_1 m_1 J_2 m_2 \,|\, J_3 m_3)\, {}^{\kappa_3}D^{J_3}_{m_3\mu_3}(R)\ , \tag{A19}$$

where the first factor in the right-hand side is a new type of Clebsch-Gordan coefficient (complex in general) which is related to the conventional Clebsch-Gordan coefficients via, from (2.3),

$$\begin{aligned}\langle \kappa_1 J_1 m_1 \kappa_2 J_2 m_2 \,|\, \kappa_3 J_3 m_3\rangle = 2[&(J_1 m_1 J_2 m_2 \,|\, J_3 m_3) + \kappa_1^*(-)^{J_1-m_1}(J_1 - m_1 J_2 m_2 \,|\, J_3 m_3)\\ &+ \kappa_2^*(-)^{J_2-m_2}(J_1 m_1 J_2 - m_2 \,|\, J_3 m_3) + \kappa_3(-)^{J_3-m_3}(J_1 m_1 J_2 m_2 \,|\, J_3 - m_3)]\ \theta(m_1)\theta(m_2)\theta(m_3)\end{aligned} \tag{A20}$$

with a constraint,

$$\kappa_3 = \kappa_1\kappa_2\ . \tag{A21}$$

If this is not satisfied, the coefficient (A20) is identically zero. Using symmetry in the conventional Clebsch-Gordan coefficients, the relation (A19) can be recast into a form more useful for our purposes,

$${}^{\kappa}D^{J}_{m\mu}{}^{*}(R)\, {}^{\kappa'}D^{J'}_{m'\mu'}(R) = \sum_{LMN}\left(\frac{2L+1}{2J+1}\right)\langle \kappa' J' m' \kappa_L LM \,|\, \kappa J m\rangle\, (J'\mu' LN \,|\, J\mu)\, {}^{\kappa_L}D^{L}_{MN}{}^{*}(R) \tag{A22}$$

with $\kappa_L = \kappa\kappa'^{*}$. Combining this with (A18) we obtain

$$\int dR\, {}^{\kappa}D^{J}_{m\mu}{}^{*}(R)\, {}^{\kappa_L}D^{L}_{MN}(R)\, {}^{\kappa'}D^{J'}_{m'\mu'}(R) = \frac{8\pi^2}{2J+1}\langle \kappa' J' m' \kappa_L LM \,|\, \kappa J m\rangle\, (J'\mu' LN \,|\, J\mu)\ . \tag{A23}$$

For completeness, we give here two different sum rules for the new D functions. These relations follow from the completeness of spin states,

$$\sum_{\kappa m} {}^{\kappa}D^{J}_{m\mu'}{}^{*}(R)\, {}^{\kappa}D^{J}_{m\mu}(R) = \delta_{\mu'\mu} \tag{A24}$$

and

$$\sum_{\mu} {}^{\kappa'}D^{J}_{m'\mu}(R)\, {}^{\kappa}D^{J}_{m\mu}{}^{*}(R) = \delta_{m'm}\delta_{\kappa'\kappa}\ . \tag{A25}$$

We are now ready to give an expression for the angular distribution in terms of reflectivity density-matrix elements. It reads

$$I^{aa'}(R) = N_J N_{J'}\, {}^{\epsilon}\rho^{aa'}_{mm'}\, {}^{\kappa}D^{J}_{m\mu}{}^{*}(R)\, {}^{\kappa'}D^{J'}_{m'\mu'}(R)\, F^{a}_{\mu\lambda} F^{a'}_{\mu'\lambda}{}^{*}\ , \tag{A26}$$

where $\kappa = \epsilon\eta$ and $\kappa' = \epsilon\eta'$, and summation over ϵ is implied. From (A23) we obtain the moment appropriate for this distribution,

$$\begin{aligned}{}^{\kappa_L}H^{aa'}(LMN) &= \langle {}^{\kappa_L}D^{L}_{MN}(R)\rangle\\ &= {}^{\kappa_L}t^{aa'}_{LM}\, G^{aa'}_{LN}\ ,\end{aligned} \tag{A27}$$

where the second factor is the same as in (A12) and

$${}^{\kappa_L}t^{aa'}{}^{*}_{LM} = \left(\frac{2J'+1}{2J+1}\right)^{1/2}\sum_{\epsilon\, mm'} {}^{\epsilon}\rho^{aa'}_{mm'}\, \langle \kappa' J' m' \kappa_L LM \,|\, \kappa J m\rangle \tag{A28}$$

with $\kappa_L = \kappa\kappa'^{*} = \eta\eta'$. It can be shown that the above quantity is simply proportional to the conventional multipole parameters, viz.,

$${}^{\kappa_L}t^{aa'}_{LM} = 2\,\theta(M)\, t^{aa'}_{LM}\ , \tag{A29a}$$

so that

$${}^{\kappa_L}H^{aa'}(LMN) = 2\,\theta(M)\, H^{aa'}(LMN)\ . \tag{A29b}$$

In terms of these moments the angular distributions (A26) assume the form

$$I^{aa'}(R) = \sum_{LM}\left(\frac{2L+1}{8\pi^2}\right){}^{\kappa_L}H^{aa'}(LMN)\, {}^{\kappa_L}D^{L}_{MN}{}^{*}(R)\ , \tag{A30}$$

where $\kappa_L = \eta\eta'$. Note that substitution of (A29b) and (A17) into (A30) brings back the angular distribution (A26).

Finally, we discuss briefly the measurability of ρ. Let us consider for the sake of simplicity the decay of R into $(s+\pi)$. Under certain circumstances,[17] the phase of F^{a}_{λ} does not depend on λ. Then, if the overall phase of F^{a}_{λ} is absorbed into $\rho^{aa'}_{mm'}$, we can set F^{a}_{λ} to be real without loss of generality. In this case, it can be shown from parity conservation in both the production and the decay of R that

$$\begin{aligned}I^{aa'}(\Omega) + I^{a'a}(\Omega) = 4\pi N_J N_{J'}\, &\mathrm{Re}\rho^{aa'}_{mm'}\, \mathrm{Re}[D^{J}_{m\lambda}{}^{*}(\Omega)\, D^{J'}_{m'\lambda}(\Omega)]\\ &\times F^{a}_{\lambda} F^{a'}_{\lambda}\ ,\end{aligned} \tag{A31}$$

so that only the real part of $\rho^{aa'}_{mm'}$ can be measured. Note that F^q_λ can be considered real in general if $s = 0$ or $s = \frac{1}{2}$. Therefore, in these cases one can determine only the real part of ρ. Since ϵ is real for bosons, it follows that only the real part of ${}^{\epsilon}\rho^{aa'}_{mm'}$ can be measured; however, this does not apply to fermions, since ϵ is purely imaginary.

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