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General Spin-Density Matrix
—Version II—

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Abstract

A derivation is given of the symmetry properties of the general spin-density matrix arising from the process $ab \to cd$, where the participating particles are arbitrary and include baryons as well as photons. A general discussion of the transformation properties of massive- as well as massless-particle states are given in the appendices.

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1 Introduction

Here we derive the symmetry properties of the general spin-density matrix for a system c produced in

$$a + b \to c + d \tag{1}$$

where the participating particles are arbitrary and include baryons as well as photons. We assume that c is an intermediate state, mesonic or baryonic, and it couples to any number of allowed decay channels.

For background material, the reader may consult the author's work on spin formalisms[**?**]. The density matrices in the reflectivity basis were given by Chung and Trueman[**?**]. For a deeper understanding of the quantum treatment of one-particle states, the reader is referred to a book by Weinberg[**?**].

Appendix A deals with the transformation properties of the reflection—as well as the parity—operator on the helicity and canonical states for massive particles, i.e. those whose mass is greater than zero[**?**]. A general discussion of the transformation properties for massless-particle states are given in Appendix B.

2 General Angular Distributions

Let $|jm\rangle$ be the spin state for c where the quantization axis is defined in the production plane, i.e. one takes either the helicity or the Jackson rest frame for c. The amplitude for production and decay of the c is

$$A \propto \sum_{\chi m} \langle \vec{p}_c\, \chi m,\, \vec{p}_d \lambda_d\, |T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle D^{\chi}_{m}(\tau) \tag{2}$$

where χ specifies the complete quantum state c which includes its spin j, its parity, its charge conjugation, its isotopic spin, and its decay products with the phase-space element given by τ; λ's refer to the helicities; T is the transition operator of the process $ab \to cd$; and D is the decay amplitude for c. We have suppressed, for brevity, the spins s_a, s_b and s_d for particles a, b and d. The distribution function follows immediately

$$I(\tau) \propto \sum_{\substack{\chi m \\ \chi' m' \\ \lambda_a \lambda_b \lambda_d}} \langle \vec{p}_c\, \chi m,\, \vec{p}_d \lambda_d\, |T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle \langle \vec{p}_c\, \chi' m',\, \vec{p}_d \lambda_d\, |T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle^* D^{\chi}_{m}(\tau)\, D^{\chi'\,*}_{m'}(\tau) \tag{3}$$

Note that the helicities of a, b and d are the 'external' unobserved variables and therefore summed over outside of the absolute square of the amplitude A. In terms of the generalized spin-density matrix

$$\rho^{\chi\chi'}_{mm'} \propto \sum_{\lambda_a \lambda_b \lambda_d} \langle \vec{p}_c\, \chi m,\, \vec{p}_d \lambda_d\, |T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle \langle \vec{p}_c\, \chi' m',\, \vec{p}_d \lambda_d\, |T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle^* \tag{4}$$

the distribution function assumes an elegant form

$$I(\tau) \propto \sum_{\substack{\chi m \\ \chi' m'}} \rho^{\chi\,\chi'}_{mm'}\, D^{\chi}_{m}(\tau)\, D^{\chi'\,*}_{m'}(\tau) \tag{5}$$

We are now ready to introduce the reflection operator through the production plane for $ab \to cd$. Let this plane be defined to be the x-z plane, i.e. the production normal is along the y-axis. Then the reflection operator defined by

$$\Pi_y = U[R_y(\pi)]\,\Pi = \Pi\,U[R_y(\pi)] \tag{6}$$

where $U[R_y(\pi)]$ is a unitary operator representing a rotation by π around the y-axis, i.e.

$$U[R_y(\pi)] = \exp(-i\pi\, J_y) \tag{7}$$

which is simply given by the standard d-function

$$U_{m'\,m}[R_y(\pi)] = d^{j}_{m'\,m}(\pi) = (-)^{j-m}\delta_{m',\,-m} \tag{8}$$

Note also that

$$U_{m'\,m}[R_y(-\pi)] = d^{j}_{m'\,m}(-\pi) = (-)^{j+m}\delta_{m',\,-m} \tag{9}$$

The reflection operator acts on a ket state $|\chi\, m\rangle$ as follows:

$$\Pi_y\,|\vec{p}_c\,\chi\,m\rangle = \eta_c\,(-)^{j-m}\,|\vec{p}_c\,\chi\;-m\rangle \tag{10}$$

where η_c is the intrinsic parity of the c (see Appendix A). Note that

$$\Pi_y^2 = (-)^{2j}\,I \tag{11}$$

where I is the identity operator. Likewise, the helicity states for a, b and d transform under the reflection operator

$$\Pi_y\,|\vec{p}\,s\,\lambda\rangle = \eta\,(-)^{s-\lambda}\,|\vec{p}\,s\;-\lambda\rangle \tag{12}$$

suppressing indices corresponding to a, b and d. Again see Appendix A.

So far we have assumed that the states a, b, c and d refer to massive particles. We assume c is always massive, since there exist no known massless-particle decays—strong or otherwise. It is shown in Appendix B that the massless ket states transform in exactly the same way [see (**??**)] as do the massive-particle states—but with one proviso that the helicity λ is restricted to two values, $+s$ and $-s$ for a massless particle (whereas it ranges over $2s+1$ values for a massive particle).

We now move over to the reflection-basis states for c, i.e.

$$|\vec{p}_c\,\epsilon\,\chi\,m\rangle = \theta(m)\left\{|\vec{p}_c\,\chi\,m\rangle + \epsilon\,\eta_c\,(-)^{j-m}\,|\vec{p}_c\,\chi\;-m\rangle\right\} \tag{13}$$

where

$$\begin{aligned} \theta(m) &= \frac{1}{\sqrt{2}}, \qquad m > 0 \\ &= \frac{1}{2}, \qquad m = 0 \\ &= 0, \qquad m < 0 \end{aligned} \tag{14}$$

These basis states constitute eigenvectors of the reflection operator

$$\Pi_y \, |\vec{p}_c \, \epsilon \, \chi \, m\rangle = \epsilon \, (-)^{2j} \, |\vec{p}_c \, \epsilon \, \chi \, m\rangle \tag{15}$$

The relationship (**??**) requires that $\epsilon^2 = (-)^{2j}$, so that $\epsilon = \pm 1$ for bosons and $\epsilon = \pm i$ for fermions. Note, however, that $\epsilon\epsilon^* = +1$ in general.

The generalized density matrix in the reflectivity basis is

$$^{\epsilon\,\epsilon'}\rho^{\chi\,\chi'}_{mm'} \propto \sum_{\lambda_a\lambda_b\lambda_d} \langle \epsilon\,\vec{p}_c\,\chi m,\, \vec{p}_d\lambda_d\,|T|\,\vec{p}_a\lambda_a,\, \vec{p}_b\lambda_b\rangle \langle \epsilon'\,\vec{p}_c\,\chi' m',\, \vec{p}_d\lambda_d\,|T|\,\vec{p}_a\lambda_a,\, \vec{p}_b\lambda_b\rangle^* \tag{16}$$

We shall explore the consequences of a reflection operation applied to the transition matrices above. The key observation is that Π_y leaves the transition operator T unperturbed, i.e. $[\Pi_y, T] = 0$. The three-vectors $\vec{p}_i$ ($i = a, b, c, d$) are left unchanged under Π_y, i.e. the boost operators and/or the rotations about the y-axis, which enter in the definitions of the helicity or the Jackson frames, remain invariant under Π_y. Therefore, the parity conservation is relegated to exploring the consequences of Π_y acting on the '<u>rest</u>' states. Insert ${\Pi_y}^{-1}\,\Pi_y = I$ next to each T in (**??**) and propagate $\Pi_y^\dagger$ and Π_y backwards and forwards, respectively, to find

$$\begin{aligned} ^{\epsilon\,\epsilon'}\rho^{\chi\,\chi'}_{mm'} \propto \epsilon\epsilon'^{\,*}\,(-)^{2(j-j')} \sum_{\lambda_a\lambda_b\lambda_d} & \langle \epsilon\,\vec{p}_c\,\chi m, \vec{p}_d, -\lambda_d\,|T|\,\vec{p}_a, -\lambda_a,\, \vec{p}_b, -\lambda_b\rangle \\ & \langle \epsilon'\,\vec{p}_c\,\chi' m',\, \vec{p}_d, -\lambda_d\,|T|\,\vec{p}_a, -\lambda_a,\, \vec{p}_b, -\lambda_b\rangle^* \end{aligned} \tag{17}$$

so that

$$^{\epsilon\,\epsilon'}\rho^{\chi\,\chi'}_{mm'} = \epsilon\epsilon'^{\,*} \times \,^{\epsilon\,\epsilon'}\rho^{\chi\,\chi'}_{mm'} \tag{18}$$

So we conclude that $\epsilon = \epsilon'$. Note that we have carefully handled the derivation, so that it applies to both bosons and fermions simultaneously. The density matrix can be written, quite generally,

$$^{\epsilon}\rho^{\chi\,\chi'}_{mm'} \propto \sum_{\lambda_a\lambda_b\lambda_d} \langle \epsilon\,\vec{p}_c\,\chi m,\, \vec{p}_d\lambda_d\,|T|\,\vec{p}_a\lambda_a,\, \vec{p}_b\lambda_b\rangle \langle \epsilon\,\vec{p}_c\,\chi' m',\, \vec{p}_d\lambda_d\,|T|\,\vec{p}_a\lambda_a,\, \vec{p}_b\lambda_b\rangle^* \tag{19}$$

a fundamental formula which incorporates parity conservation in the production process $ab \to cd$. The distribution function in the reflectivity basis is

$$I(\tau) \propto \sum_{\epsilon} \sum_{\substack{\chi m \\ \chi' m'}} {}^{\epsilon}\rho^{\chi\,\chi'}_{mm'}\ {}^{\epsilon}D^{\chi}_{m}(\tau)\ {}^{\epsilon}D^{\chi'\,*}_{m'}(\tau) \tag{20}$$

where $^{\epsilon}D$ is the decay amplitude in the reflectivity basis.

The rank of the density matrix is determined by the number of independent terms in the summation on helicities, see (**??**). Let

$$n_i = 2s_i + 1 \quad \text{or} \quad n_i = 2 \quad \text{for} \quad i = a,\ b,\ d$$

depending on whether a particle is massive or massless. The total number in the sum in (**??**) is

$$N = n_a\, n_b\, n_d \tag{21}$$

So the rank of the density matrix is $(N+1)/2$ if N is odd, and it is $N/2$ if N is even. Note that the reduction in the rank comes from parity conservation in the production process. We give in Table I a few illustrative examples.

Table I. Rank of Spin-Density Matrix for X

Reaction	Rank
$\pi^- p \to \pi^- X^+$	1
$\pi^- p \to X^- p$	2
$\pi^+ n \to X^- \Delta^{++}$	4
$\bar{p} p \to X^- p$	4
$\bar{n} p \to X^- \Delta^{++}$	8
$\gamma p \to X^0 p$	4
$\gamma p \to \pi^+ X^0$	2
$\nu p \to e^- X^{++}$	$1^{\dagger}$
$e^- p \to e^- X^+$	$1^{\dagger}$
$\phi p \to X^0 p$	6
$\phi p \to \pi^+ X^0$	3
$\pi^- \eta \to \pi^- X^0$	1
$\pi^- \phi \to \pi^- X^0$	2

† The electrons are assumed to come with one helicity—true in general for an odd-half-integer spin in the limit of zero mass.

Acknowledgment

I am indebted to T. L. Trueman for enlightening discussions on the properties of massless-particle states.

Appendix A

We work out here the transformation properties of the ket states of massive particles. The effect of parity operation on a general helicity state is non-trivial; the transformation law is given here *for the first time,* following the derivation given by Weinberg[**?**] for massless-particle states. We shall use a convention in which p (or k) stands for the four-momentum and the magnitude of the three-momentum, of a massive particle.

We define as the 'proto' ket state of a massive particle as that in its rest frame with its four-momentum $k = (w, 0, 0, 0)$, where w is the mass of the particle with $w > 0$. The ket state $|k, \lambda\rangle$ transform according to the standard $|jm\rangle$ representation of the angular momentum operators J_x, J_y and J_z. The parity operation on the 'proto' ket state induces a transformation, noting that it commutes with $\vec{J}$,

$$\Pi\,|k, \lambda\rangle = \eta\,|k, \lambda\rangle \tag{22}$$

where $\eta = \pm 1$ is the intrinsic parity. A rotation by π around the y-axis yields

$$U[R_y(\pi)]\,|k, \lambda\rangle = (-)^{j-\lambda}\,|k, -\lambda\rangle \tag{23}$$

Combining the two equations above, we see how the reflection operator Π_y around the y-axis acts on our ket state:

$$\Pi_y\,|k, \lambda\rangle = \Pi\,U[R_y(\pi)]\,|k, \lambda\rangle = U[R_y(\pi)]\,\Pi\,|k, \lambda\rangle = \eta\,(-)^{j-\lambda}\,|k, -\lambda\rangle \tag{24}$$

We now apply a boost along the z-axis, i.e. $p = (E, 0, 0, p)$,

$$B_z(p) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} \tag{25}$$

where $\tanh\alpha = \beta = p/E$, $\cosh\alpha = \gamma = E/w$ and $\sinh\alpha = \gamma\beta = p/w$, with the Lorentz metric

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = +1$$

A boost operator must satisfy

$$B^{\mu}{}_{\alpha}\; g_{\mu\nu}\; B^{\nu}{}_{\beta} = g_{\alpha\beta} \tag{26}$$

a condition for a Lorentz transformation. Note that

$$p^{\mu} = [B_z(p)]^{\mu}{}_{\nu}\,k^{\nu} \tag{27}$$

and that the corresponding ket states $|p, \lambda\rangle$ are given by

$$|p, \lambda\rangle = U[B_z(p)]\,|k, \lambda\rangle \tag{28}$$

We now apply $U[B_z(p)]$ to both sides of (**??**), to find

$$\Pi_y\,|p, \lambda\rangle = \eta\,(-)^{j-\lambda}\,|p, -\lambda\rangle \tag{29}$$

since the reflection operator Π_y and $U[B_z(p)]$ commute.

Let $R_y(\theta)$ be the rotation operator which takes $p = (E, 0, 0, p)$ to

$$\mathring{p} = (E, \vec{p}) = (E, p\sin\theta, 0, p\cos\theta)$$

i.e. the $\vec{p}$ is a vector confined to the x-z plane. We see that

$$R_y(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{30}$$

and

$$\overset{\circ}{p}^{\mu} = [R_y(\theta)]^{\mu}{}_{\nu}\, p^{\nu} \tag{31}$$

with the corresponding unitary operator given by

$$U[R_y(\theta)] = \exp(-i\,\theta\, J_y) \tag{32}$$

We define the helicity ket state $|\vec{p},\lambda\rangle$ by

$$|\vec{p},\lambda\rangle = U[R_y(\theta)\, B_z(p)]\, |k,\lambda\rangle \tag{33}$$

with the normalization given by

$$\langle\vec{p},\lambda|\vec{p}\,',\lambda'\rangle = 2E\,(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}\,')\,\delta_{\lambda\,\lambda'} \tag{34}$$

assuming the same spin j for both states. But $U[R_y(\theta)]$ and $U[B_z(p)]$ commute with Π_y, so that

$$\Pi_y\,|\vec{p},\lambda\rangle = \eta\,(-)^{j-\lambda}\,|\vec{p},-\lambda\rangle \tag{35}$$

This gives the transformation law of a reflection operator through the y-axis on a helicity state whose momentum lies in the x-z plane, i.e. $p_y = 0$.

We can readily work out the transformation properties of a reflection operator through the y-axis on a canonical state whose momentum lies in the x-z plane. Recall that a canonical state[a] in the x-z plane is given by

$$|\vec{p},jm\rangle = U[R_y(\theta)]\,U[B_z(p)]\,U^{-1}[R_y(\theta)]\,|jm\rangle \tag{36}$$

with the normalization given by

$$\langle\vec{p},jm|\vec{p}\,',j'm'\rangle = 2E\,(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}\,')\,\delta_{j\,j'}\,\delta_{m\,m'} \tag{37}$$

Note that $|jm\rangle$ is the rest-frame ket state identical to $|k,\lambda\rangle$ with $m=\lambda$. A reflection operator commutes with both a rotation around the y-axis and a boost along the z-axis, so that

$$\Pi_y\,|\vec{p},jm\rangle = \eta\,(-)^{j-m}\,|\vec{p},j,-m\rangle \tag{38}$$

identical to (**??**). Note that

$$\Pi_y^2 = (-)^{2j}\,I, \qquad \Pi_y^{-1} = (-)^{2j}\,\Pi_y \tag{39}$$

The formula (**??**) is the transformation law appropriate for a state defined, e.g. in the Jackson frame.

Finally, we go over to a discussion of the effect of a general homogeneous Lorentz transformation. In order to proceed with the discussion, we have to first define general helicity and

[a] We distinguish a canonical state from a helicity state by two conventions: a helicity is denoted by λ, whereas the canonical z-component of spin is denoted by m; and a helicity state has the spin j suppressed, while it is explicitly given for a canonical state.

canonical states, for which the direction of the momentum $\hat{p}$ is arbitrary. For the purpose, define

$$R(\hat{p}) = R_z(\phi)\, R_y(\theta) \tag{40}$$

where $R_y(\theta)$ is as given in (**??**) and $R_z(\phi)$ is given by

$$R_z(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{41}$$

so that

$$\overset{\circ}{p}{}^{\mu} = [R(\hat{p})]^{\mu}{}_{\nu}\, p^{\nu} \tag{42}$$

We define

$$\overset{\circ}{p} = (E, \vec{p}) = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta) \tag{43}$$

while we reserve the notation of the four-momentum p to mean $p = (E, 0, 0, p)$. The corresponding unitary operator is, as usual,

$$U[R(\hat{p})] = U[R_z(\phi)R_y(\theta)] = \exp(-i\,\phi\, J_z)\, \exp(-i\,\theta\, J_y) \tag{44}$$

So the most general helicity and canonical states are defined by

$$\begin{aligned} |\vec{p}, \lambda\rangle &= U[\mathcal{L}(\overset{\circ}{p})]\, |k, \lambda\rangle = U[R(\hat{p})]\, U[B_z(p)]\, |k, \lambda\rangle = U[L(\overset{\circ}{p})]\, U[R(\hat{p})]\, |k, \lambda\rangle \\ |\vec{p}, jm\rangle &= U[L(\overset{\circ}{p})]\, |jm\rangle = U[R(\hat{p})]\, U[B_z(p)]\, U^{-1}[R(\hat{p})]\, |jm\rangle \end{aligned} \tag{45}$$

where

$$\begin{aligned} U[\mathcal{L}(\overset{\circ}{p})] &= U[R(\hat{p})]\, U[B_z(p)] \\ U[L(\overset{\circ}{p})] &= U[R(\hat{p})]\, U[B_z(p)]\, U^{-1}[R(\hat{p})] \end{aligned} \tag{46}$$

The second formula above has been used to obtain the first formula of (**??**)—this in turn shows that the helicity and canonical states are related through

$$|\vec{p}, \lambda\rangle = \sum_m D^j_{m\lambda}(\phi, \theta, 0) |\vec{p}, jm\rangle \tag{47}$$

The inverse is

$$|\vec{p}, jm\rangle = \sum_\lambda D^{j\,*}_{m\lambda}(\phi, \theta, 0)\, |\vec{p}, \lambda\rangle \tag{48}$$

from orthonormality of the D-functions.

Let Λ be a general homogeneous Lorentz transformation, i.e.

$$p'^{\,\mu} = \Lambda^{\mu}{}_{\nu}\, \overset{\circ}{p}{}^{\nu}$$

Its action on our ket states is

$$
\begin{aligned}
U(\Lambda)\,|\vec{p},\lambda\rangle &= U(\Lambda)\,U[\mathcal{L}(\mathring{p})]\,|k,\lambda\rangle \\
&= U[\mathcal{L}(\Lambda\mathring{p})]\,U[\mathcal{L}^{-1}(\Lambda\mathring{p})\,\Lambda\,\mathcal{L}(\mathring{p})]\,|k,\lambda\rangle \\
U(\Lambda)\,|\vec{p},jm\rangle &= U(\Lambda)\,U[L(\mathring{p})]\,|jm\rangle \\
&= U[L(\Lambda\mathring{p})]\,U[L^{-1}(\Lambda\mathring{p})\,\Lambda\,L(\mathring{p})]\,|jm\rangle
\end{aligned}
\tag{49}
$$

But the second operators above leave the rest-frame ket states invariant (the second of these is known as the Wigner rotation), so we obtain

$$
\begin{aligned}
U(\Lambda)\,|\mathring{p},\lambda\rangle &= \sum_{\lambda'} D^j_{\lambda'\lambda}[\mathcal{L}^{-1}(\Lambda\mathring{p})\,\Lambda\,\mathcal{L}(\mathring{p})]\,|\Lambda\mathring{p},\lambda'\rangle \\
U(\Lambda)\,|\mathring{p},jm\rangle &= \sum_{m'} D^j_{m'm}[L^{-1}(\Lambda\mathring{p})\,\Lambda\,L(\mathring{p})]\,|\Lambda\mathring{p},jm'\rangle
\end{aligned}
\tag{50}
$$

where D^j is the standard rotation function. A special and practically important case arises when Λ represents a pure rotation $\mathcal{R}$. We see that, from (**??**),

$$
\begin{aligned}
&U(\mathcal{R})\,U[\mathcal{L}(\mathring{p})] = U[\mathcal{R}R(\hat{p})]\,U[B_z(p)] \\
&U(\mathcal{R})\,U[L(\mathring{p})] = U[\mathcal{R}R(\hat{p})]\,U[B_z(p)]\,U^{-1}[\mathcal{R}R(\hat{p})]\,U(\mathcal{R})
\end{aligned}
\tag{51}
$$

where the second equation above has been obtained by multiplying it with $U^{-1}(\mathcal{R})\,U(\mathcal{R}) = I$ from the right. From (**??**), (**??**) and (**??**), we find

$$
\begin{aligned}
U(\mathcal{R})\,|\vec{p},\lambda\rangle &= |\mathcal{R}\,\vec{p},\lambda\rangle = |\,\vec{p}\,',\lambda\rangle \\
U(\mathcal{R})\,|\vec{p},jm\rangle &= \sum_{m'} D^j_{m'm}(\mathcal{R})|\mathcal{R}\,\vec{p},jm'\rangle = \sum_{m'} D^j_{m'm}(\mathcal{R})|\,\vec{p}\,',jm'\rangle
\end{aligned}
\tag{52}
$$

The first relationship above shows that helicity is conserved under a pure rotation; the second shows that canonical states transform as if they were rest-frame states. This means that all of the non-relativistic formalism can be applied to the problems involving relativistic particles—but with a certain caveat[**?**].

Finally, we work out the general transformation law of a parity operation on helicity and canonical states. We see that, from (**??**),

$$
\begin{aligned}
\Pi\,|\vec{p},\lambda\rangle &= \Pi\,U[R(\hat{p})B_z(p)]\,|k,\lambda\rangle \\
&= U[R(\hat{p})]\,U^{-1}[R_y(\pi)]\,U[R_y(\pi)]\,\Pi\,U[B_z(p)]\,|k,\lambda\rangle \\
&= \eta\,(-)^{j-m}\,U[R(\hat{p})R_y(-\pi)B_z(p)]\,|k,-\lambda\rangle
\end{aligned}
\tag{53}
$$

From (**??**) we may define

$$
U[R(-\hat{p})] = U[R_z(\phi\pm\pi)R_y(\pi-\theta)] = \exp[-i\,(\phi\pm\pi)\,J_z]\,\exp[-i\,(\pi-\theta)\,J_y]
\tag{54}
$$

where $\phi + \pi$ $(\phi - \pi)$ corresponds to $0 \leq \phi < \pi$ $(\pi \leq \phi < 2\pi)$. Now, observe

$$\begin{aligned}
&U^{-1}[R(-\hat{p})]\, U[R(\hat{p})]\, U[R_y(-\pi)] \\
&\qquad = \exp[+i\,(\pi - \theta)\, J_y]\, \exp[+i\,(\phi \pm \pi)\, J_z] \\
&\qquad \times\, \exp(-i\,\phi\, J_z)\, \exp(-i\,\theta\, J_y)\, \exp(+i\,\pi\, J_y) \\
&\qquad = \exp[+i\,(\pi - \theta)\, J_y]\, \exp(\pm i\,\pi\, J_z)\, \exp[+i\,(\pi - \theta)\, J_y] \\
&\qquad = \exp(\pm i\,\pi\, J_z)
\end{aligned} \tag{55}$$

since a rotation of $\pm\pi$ around the z-axis reverses the sign of J_y. But a rotation around the z-axis commutes with a boost along the same axis, so that

$$U[R(\hat{p}) R_y(-\pi) B_z(p)] = U[R(-\hat{p}) B_z(p)]\, \exp(\pm i\,\pi\, J_z) \tag{56}$$

Introducing a ket state $|-\vec{p}, \lambda\rangle$ defined by

$$|-\vec{p}, \lambda\rangle = U[R(-\hat{p}) B_z(p)]\, |k, \lambda\rangle \tag{57}$$

we obtain

$$\Pi\, |\vec{p}, \lambda\rangle = \eta\, (-)^{j-\lambda}\, \exp(\mp i\,\pi\,\lambda)\, |-\vec{p}, -\lambda\rangle \tag{58}$$

with the phase $-i\,\pi\,\lambda$ or $+i\,\pi\,\lambda$ according to whether $p_y \geq 0$ or $p_y < 0$, respectively. We see that

$$\begin{aligned}
\Pi\, |-\vec{p}, \lambda\rangle &= \Pi\, U[R(-\hat{p}) B_z(p)]\, |k, \lambda\rangle \\
&= U[R(-\hat{p})]\, U^{-1}[R_y(\pi)]\, U[R_y(\pi)]\, \Pi\, U[B_z(p)]\, |k, \lambda\rangle \\
&= \eta\, (-)^{j-\lambda}\, U[R(-\hat{p}) R_y(-\pi) B_z(p)]\, |k, -\lambda\rangle
\end{aligned} \tag{59}$$

but

$$\begin{aligned}
&U^{-1}[R(\hat{p})]\, U[R(-\hat{p})]\, U[R_y(-\pi)] \\
&\qquad = \exp(+i\,\theta\, J_y)\, \exp(+i\,\phi\, J_z) \\
&\qquad \times\, \exp[-i\,(\phi \pm \pi)\, J_z]\, \exp[-i\,(\pi - \theta)\, J_y]\, \exp(+i\,\pi\, J_y) \\
&\qquad = \exp(+i\,\theta\, J_y)\, \exp(\mp i\,\pi\, J_z)\, \exp(+i\,\theta\, J_y) \\
&\qquad = \exp(\mp i\,\pi\, J_z)
\end{aligned} \tag{60}$$

and so

$$U[R(-\hat{p}) R_y(-\pi) B_z(p)] = U[R(\hat{p}) B_z(p)]\, \exp(\mp i\,\pi\, J_z) \tag{61}$$

The parity operation on $|-\vec{p}, \lambda\rangle$ induces a transformation

$$\Pi\, |-\vec{p}, \lambda\rangle = \eta\, (-)^{j-\lambda}\, \exp(\pm i\,\pi\,\lambda)\, |\vec{p}, -\lambda\rangle \tag{62}$$

with the phase $+i\,\pi\,\lambda$ or $-i\,\pi\,\lambda$ according to whether $p_y \geq 0$ or $p_y < 0$, respectively. Note that the formula (**??**) can be obtained by substituting $\vec{p} \to -\vec{p}$ in (**??**) and observing that p_y undergoes a sign change. We see that, combining (**??**) and (**??**),

$$\Pi^2 = I, \qquad \Pi^{-1} = \Pi \tag{63}$$

as expected.

The transformation law of a parity operation is much simpler for canonical one-particle states. Observe that

$$\Pi\, U[B_z(p)]\, \Pi^{-1} = U[B_z(\bar{p})] \tag{64}$$

where $p = (E, 0, 0, p)$ and $\bar{p} = (E, 0, 0, -p)$, so that

$$\begin{aligned} \Pi\, U[L(\mathring{p})]\, \Pi^{-1} &= U[R(\hat{p})]\, U[B_z(\bar{p})]\, U^{-1}[R(\hat{p})] \\ &= U[R(-\hat{p})]\, U[B_z(p)]\, U^{-1}[R(-\hat{p})] \end{aligned} \tag{65}$$

We define, using a notation $\breve{p} = (E, -\vec{p}\,)$ to distinguish it from $\mathring{p} = (E, \vec{p}\,)$,

$$| -\vec{p}, jm\rangle = U[L(\breve{p})]\, |jm\rangle \tag{66}$$

where

$$U[L(\breve{p})] = U[R(-\hat{p})]\, U[B_z(p)]\, U^{-1}[R(-\hat{p})] \tag{67}$$

We may now work out the relationship between the helicity and canonical states with $-\vec{p}$. From (**??**), we see that

$$| -\vec{p}, \lambda\rangle = \sum_m D^j_{m\lambda}(\phi \pm \pi, \pi - \theta, 0)\, | -\vec{p}, jm\rangle \tag{68}$$

where $\phi + \pi$ ($\phi - \pi$) corresponds to $0 \le \phi < \pi$ ($\pi \le \phi < 2\pi$). We shall find it useful to invert the formula above with $\lambda \to -\lambda$

$$\begin{aligned} | -\vec{p}, jm\rangle &= (-)^{j+m}\, \exp(\pm i\, m\, \pi) \sum_\lambda D^{j\,*}_{m\lambda}(\phi, \theta, 0)\, | -\vec{p}, -\lambda\rangle \\ &= \sum_\lambda (-)^{j-\lambda}\, \exp(\mp i\, \pi\, \lambda)\, D^{j\,*}_{m\lambda}(\phi, \theta, 0)\, | -\vec{p}, -\lambda\rangle \end{aligned} \tag{69}$$

where the second equation above has been obtained by inserting

$$\exp(\mp i\, \pi\, \lambda)\;\; \exp(\pm i\, \pi\, \lambda) = +1$$

inside the summation.

Note that, from (**??**) and (**??**),

$$\begin{aligned} \Pi\, U[L(\breve{p})]\, \Pi^{-1} &= U[R(-\hat{p})]\, U[B_z(\bar{p})]\, U^{-1}[R(-\hat{p})] \\ &= U[R(\hat{p})]\, U[B_z(p)]\, U^{-1}[R(\hat{p})] \end{aligned} \tag{70}$$

We see immediately that, using $\Pi^{-1}\,\Pi = I$, (**??**) and (**??**),

$$\begin{aligned} \Pi\, |\vec{p}, jm\rangle &= \eta\, | -\vec{p}, jm\rangle \\ \Pi\, | -\vec{p}, jm\rangle &= \eta\, |\vec{p}, jm\rangle \end{aligned} \tag{71}$$

The relationship (**??**) follows trivially from the equations above.

We now know how the helicity and canonical states transform under parity. One may ask a question: are they consistent? In order to answer this question, we start out with (**??**) and insert (**??**)

$$
\begin{aligned}
\Pi\,|\vec{p},\lambda\rangle &= \eta \sum_m D^j_{m\lambda}(\phi,\theta,0)\,|-\vec{p},jm\rangle \\
&= \eta \sum_m D^j_{m\lambda}(\phi,\theta,0) \sum_{\lambda'} (-)^{j-\lambda'} \exp(\mp i\,\pi\,\lambda')\, D^{j\,*}_{m\lambda'}(\phi,\theta,0)\,|-\vec{p},-\lambda'\rangle \\
&= \eta\,(-)^{j-\lambda} \exp(\mp i\,\pi\,\lambda)\,|-\vec{p},-\lambda\rangle
\end{aligned}
\tag{72}
$$

obtaining the result (**??**). Observe that we have obtained this relationship by an altogether different method.[b]

Appendix B

Here we work out the transformation properties of the ket state corresponding to massless particles, as given by Weinberg[**?**]. We retrace his derivation, since our convention is different from his on three counts: (a) our Lorentz metric is different; (b) our normalization of the ket states are different; and (c) we use 'active' rather than 'passive' rotations and Lorentz transformations. We shall use a convention in which p (or k) stands for four-momentum, magnitude of the three-momentum and energy, for a massless particle.

We define as the 'proto' ket state for a massless particle as that with its four-momentum $k = (k, 0, 0, k)$. It can be written $|k, \lambda\rangle$ with the helicity defined through

$$J_z\,|k,\lambda\rangle = \lambda\,|k,\lambda\rangle \tag{73}$$

where J_z is the angular momentum operator along the z-axis. The normalization is given by

$$\langle k,\lambda|k',\lambda'\rangle = 2k\,(2\pi)\,\delta(k-k')\,\delta_{\lambda\,\lambda'} \tag{74}$$

A general massless state can now be defined via

$$|\vec{p},\lambda\rangle = U[R_z(\phi)R_y(\theta)B_z(p/k)]\,|k,\lambda\rangle \tag{75}$$

where $B_z(u)$ is the boost operator along the z-axis given by

$$B_z(u) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} \tag{76}$$

[b] This is the method used in Ref [**?**], but the formula corresponding to (**??**) contains a factor $\exp(-i\,\pi\,j)$, instead of the factor $(-)^{j-\lambda}\exp(\mp i\,\pi\,\lambda)$ given in this note. Note that, if only one sign had been used, i.e. $\exp(-i\,\pi\,\lambda)$, which means that ϕ is now allowed to be greater than 2π, then we would have obtained again the same factor $\exp(-i\,\pi\,j)$ given in Ref [**?**].

and $\alpha = \ln(u)$ with $u = p/k$. The rotation $R(\hat{p}) = R_z(\phi)\, R_y(\theta)$ has already been specified by (**??**) and (**??**). The normalization is given by

$$\langle \vec{p}, \lambda | \vec{p}\,', \lambda' \rangle = 2p\,(2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{p}\,')\, \delta_{\lambda\,\lambda'} \tag{77}$$

assuming the same spin j for both states.

The parity operator Π induces a transformation in our 'proto' state $|k, \lambda\rangle$ such that $k = (k, 0, 0, k)$ is turned into $\bar{k} = (k, 0, 0, -k)$ and λ is unperturbed because Π commutes with $\vec{J}$, so that

$$\Pi\, |k, \lambda\rangle \propto |\bar{k}, -\lambda\rangle$$

The change in sign for λ results from the fact that it represents the helicity, and it has to change sign because the momentum has flipped. Since Π_y leaves k invariant but reverses the sign of J_z, one must have

$$\Pi_y\, |k, \lambda\rangle = f^j_\lambda\, |k, -\lambda\rangle \tag{78}$$

where we define $\Pi_y = U[R_y(\pi)]\, \Pi$. Now one can discern the action of parity on a general ket state defined in (**??**). For the purpose, we insert an identity

$$U[R_y(-\pi)] U[R_y(\pi)] = I$$

to find

$$\begin{aligned} \Pi\, |\vec{p}, \lambda\rangle &= \Pi\, U[R_z(\phi) R_y(\theta) B_z(p/k)]\, |k, \lambda\rangle \\ &= U[R_z(\phi) R_y(\theta)]\, U[R_y(-\pi)] U[R_y(\pi)]\, \Pi\, U[B_z(p/k)]\, |k, \lambda\rangle \\ &= f^j_\lambda\, U[R_z(\phi) R_y(\theta) R_y(-\pi) B_z(p/k)]\, |k, -\lambda\rangle \end{aligned} \tag{79}$$

where we have used the fact that Π and $\vec{J}$ commute and that $B_z(u)$ and Π_y commute. The equivalent of (**??**) in the massless case is

$$U[R(\hat{p}) R_y(-\pi) B_z(p/k)] = U[R(-\hat{p}) B_z(p/k)]\, \exp(\pm i\, \pi\, J_z) \tag{80}$$

so that

$$\Pi\, |\vec{p}, \lambda\rangle = f^j_\lambda\, \exp(\mp i\, \pi\, \lambda)\, | - \vec{p}, -\lambda\rangle \tag{81}$$

with the phase $-i\,\pi\,\lambda$ or $+i\,\pi\,\lambda$ according to whether $p_y \geq 0$ or $p_y < 0$, respectively.

Consider now the case of a ket state with $\vec{p}$ in the x-z plane, and apply Π_y

$$\begin{aligned} \Pi_y\, |\vec{p}, \lambda\rangle &= \Pi_y\, U[R_y(\theta) B_z(p/k)]\, |k, \lambda\rangle \\ &= U[R_y(\theta) B_z(p/k)]\, \Pi_y\, |k, \lambda\rangle \\ &= f^j_\lambda\, |\vec{p}, -\lambda\rangle \end{aligned} \tag{82}$$

where we have used the fact that Π_y commutes with any rotation around the y-axis and also with a boost along the z-axis. From the equation above, we see that

$$\Pi^2_y = f^j_\lambda\, f^j_{-\lambda} \tag{83}$$

Note also that, since the normalization of the state vectors cannot change under the operations, we must have

$$f^j_\lambda\, {f^j_\lambda}^* = +1 \tag{84}$$

One may ask how our ket states transform under Π^2. In order to work this out, we have to first understand the actions of a Π on a ket state $|-\vec{p},\lambda\rangle$:

$$\begin{aligned}\Pi\,|-\vec{p},\lambda\rangle &= U[R_z(\pi)]\,U[R_y(\pi-\theta)]\,\Pi\,B_z(p/k)]\,|k,\lambda\rangle \\ &= U[R_z(\pi)]U[R_y(\pi-\theta)]\,U[R_y(-\pi)]\,U[R_y(\pi)]\,\Pi\,B_z(p/k)]\,|k,\lambda\rangle \\ &= f^j_\lambda\,U[R_z(\pi)]U[R_y(-\theta)]\,B_z(p/k)]\,|k,-\lambda\rangle\end{aligned} \tag{85}$$

But we see that

$$\begin{aligned}U^{-1}(\hat{p})\,U[R_z(\pi)]\,U[R_y(-\theta)] &= U[R_y(-\theta)]\,U[R_z(\pi)]\,U[R_y(-\theta)] \\ &= U[R_y(-\theta)]\,U[R_y(+\theta)]\,U[R_z(\pi)] \\ &= U[R_z(\pi)]\end{aligned} \tag{86}$$

so that

$$\Pi\,|-\vec{p},\lambda\rangle = f^j_\lambda\,\exp(+i\,\pi\,\lambda)\,|\vec{p},-\lambda\rangle \tag{87}$$

From (??), we obtain, noting that $p_y = 0$,

$$\begin{aligned}\Pi^2\,|\vec{p},\lambda\rangle &= f^j_\lambda\,\exp(-i\,\pi\,\lambda)\,\Pi\,|-\vec{p},-\lambda\rangle \\ &= f^j_\lambda\,f^j_{-\lambda}\,\exp(-2\,i\,\pi\,\lambda)\,|\vec{p},\lambda\rangle\end{aligned} \tag{88}$$

and

$$\Pi^2 = f^j_\lambda\,f^j_{-\lambda}\,(-)^{2\lambda} \tag{89}$$

It is clear that the phase factor f^j_λ which satisfies (??), (??) and (??) can be written

$$f^j_\lambda = \eta\,(-)^{j-\lambda} \tag{90}$$

where $\eta = \pm 1$ is the intrinsic parity, and furthermore

$$\Pi_y^2 = (-)^{2j}\,I, \qquad \Pi^2 = I \tag{91}$$

To recapitulate, a ket state with $p_y = 0$ transforms under the reflection and parity operators according to

$$\begin{aligned}\Pi_y\,|\vec{p},\lambda\rangle &= \eta\,(-)^{j-\lambda}\,|\vec{p},-\lambda\rangle \\ \Pi\,|\vec{p},\lambda\rangle &= \eta\,(-)^{j-\lambda}\,\exp(-i\,\pi\,\lambda)\,|-\vec{p},-\lambda\rangle \\ \Pi\,|-\vec{p},\lambda\rangle &= \eta\,(-)^{j-\lambda}\,\exp(+i\,\pi\,\lambda)\,|\vec{p},-\lambda\rangle\end{aligned} \tag{92}$$

We see that the first formula is identical to that for massive-particle states [see (??)].

For completeness, we shall work out the Lorentz transformation properties of general massless ket states. It is convenient to express the Lorentz transformation which defines (??) as

$$L(\mathring{p}) = R_z(\phi)R_y(\theta)B_z(p/k) \tag{93}$$

where

$$\mathring{p} = (p,\vec{p}\,) = (p, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$$

Now let Λ be a Lorentz transformation which takes $\mathring{p}$ to p'

$$p'^{\,\mu} = \Lambda^{\mu}{}_{\nu}\, \mathring{p}^{\,\nu}$$

then we find, by a straightforward manipulation of the relevant operators,

$$\begin{aligned} U(\Lambda)\,|\mathring{p},\lambda\rangle &= U(\Lambda)U[L(\mathring{p})]\,|k,\lambda\rangle \\ &= U[L(\Lambda\mathring{p})]\,U[L^{-1}(\Lambda\mathring{p})\Lambda L(\mathring{p})]\,|k,\lambda\rangle \end{aligned} \tag{94}$$

The operator $L^{-1}(\Lambda\mathring{p})\Lambda L(\mathring{p})$ takes the standard four-vector k to $\mathring{p}$, to $\Lambda\mathring{p}$ and back to k again. So it can at most be a rotation around the z-axis which leaves $k = (k, 0, 0, k)$ invariant. Let the angle of rotation be $\Phi(\Lambda, \mathring{p})$. Then we obtain

$$U[L^{-1}(\Lambda\mathring{p})\Lambda L(\mathring{p})] = \exp[-i\,\Phi(\Lambda,\mathring{p})\,J_z] \tag{95}$$

so that

$$U(\Lambda)\,|\mathring{p},\lambda\rangle = \exp[-i\,\lambda\,\Phi(\Lambda,\mathring{p})]\,|\Lambda\mathring{p},\lambda\rangle \tag{96}$$

Therefore, the helicity is Lorentz-invariant for massless particles. Note that the helicity is *not* Lorentz-invariant for massive particles [see (**??**)], one of the fundamental differences between massive and massless particles. If Λ is a pure rotation $\mathcal{R}$, then the transformation property is simpler. Since $\vec{p}\,' = \mathcal{R}\,\vec{p}$, we see that, from (**??**),

$$U(\mathcal{R})\,|\vec{p},\lambda\rangle = |\mathcal{R}\,\vec{p},\lambda\rangle = |\,\vec{p}\,',\lambda\rangle \tag{97}$$

identical to the first eqation of (**??**), i.e. the helicity is invariant under a rotation for both massive and massless particles.

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